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Supergravity currents and linearized interactions for Matrix Theory configurations with fermionic backgrounds

Washington Taylor IV

Center for Theoretical Physics
MIT, Bldg. 6-306, Cambridge, MA 02139, U.S.A.
E-mail: wati@mit.edu

Mark Van Raamsdonk

Department of Physics, Joseph Henry Laboratories
Princeton University, Princeton, New Jersey 08544, U.S.A.
E-mail: mav@princeton.edu

Abstract: The leading terms in the long-range interaction potential between an arbitrary pair of matrix theory objects are calculated at one-loop order. This result generalizes previous calculations by including arbitrary fermionic background field configurations. The interaction potential at orders $1/r^7$ and $1/r^8$ is shown to correspond precisely with the leading terms expected from linearized supergravity interactions between arbitrary objects in M-theory. General expressions for the stress tensor, membrane current and 5-brane current of an arbitrary matrix configuration are derived, including fermionic contributions. Supergravity effects which are correctly reproduced include membrane/5-brane interactions, 0-brane/6-brane interactions, supercurrent/supercurrent interactions and the spin contributions to moments of the supergravity currents. The matrix theory description of the supergravity stress tensor, membrane current and 5-brane current are used to propose an explicit formulation of matrix theory in an arbitrary background metric and 3-form field.

Keywords: Field Theories in Lower Dimensions, M(atrix) Theories, Models of Quantum Gravity.

Contents

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1. Introduction

Over a decade ago, a simple model of supersymmetric matrix quantum mechanics [1, 2, 3] was proposed as a regulated form of the quantum supermembrane theory in 11 dimensions [4, 5, 6]. Interest in this model declined when it was found to have a continuous spectrum [7], which seemed incompatible with a first-quantized interpretation of the theory in 11D. In 1996, interest in this "Matrix Theory" was rekindled when Banks, Fischler, Shenker and Susskind (BFSS) proposed that it gives a complete description of light-front M-theory in the large N limit [8]. They pointed out that the continuous spectrum is natural since the theory should be understood as a second-quantized theory in 11 dimensions. BFSS argued that M-theory interactions, including those of classical supergravity, should arise as quantum effects in matrix theory. They gave as one piece of evidence for their conjecture the result that the leading long-range interaction between a pair of gravitons in 11D supergravity can be reproduced by a one-loop calculation in matrix theory [9].

In the two years following the BFSS conjecture, it has become clear that matrix theory encodes a remarkable amount of the structure of M-theory and 11D supergravity (for reviews, see [10, 11, 12, 13]). Matrix theory configurations corresponding to supergravitons [8, 14], membranes [6, 8] and fivebranes [15, 16] have been identified. The interactions between objects in Matrix theory have been found to agree with supergravity in a variety of situations. Two-graviton interactions have been shown to agree with supergravity up to terms of the form v^6/r^{14} [17, 18, 19]. The spin-dependent interactions between two gravitons have been checked to leading order in

$1/r$ for each power of the fermion fields encoding the graviton spin by performing one-loop matrix theory calculations [20, 21, 22, 23, 24]. It has been shown that the first nonlinear gravitational correction to 3-graviton scattering is correctly reproduced in a two-loop matrix theory calculation [25, 26, 27, 28, 29]. A subset of terms in the N-graviton interaction potential have been reproduced by an $(N-1)$-loop matrix theory calculation, although there is an indication that some 3-loop matrix theory terms may disagree with 4-graviton interactions in supergravity [29].

For more general matrix theory configurations, progress has been made towards identifying the structure responsible for the agreement between matrix theory and supergravity at leading orders. It was shown in [30] that the supergravity potential between an arbitrary pair of bosonic M-theory objects arising from the exchange of quanta with zero longitudinal momentum is exactly reproduced by terms in the one-loop matrix theory potential. This result was achieved through the identification of matrix theory quantities corresponding to the supergravity stress-energy tensor, membrane current and fivebrane current. Linearized interactions between multipole moments of the supergravity currents were shown to be reproduced correctly by an infinite series of terms in the effective matrix theory potential of the form F^4X^l/r^{7+l}.

In this work, we extend the results of [30] to a pair of completely general configurations, including non-vanishing fermionic fields in our matrix theory background. As in [30], we do not require that the matrix theory configurations preserve any supersymmetry. We demonstrate in complete generality that the on-shell matrix theory potential vanishes below $1/r^7$ and that at orders $1/r^7$ and $1/r^8$ the matrix theory calculation reproduces exactly the leading-order interactions expected from supergravity arising from the exchange of gravitons, gravitinos, and three form quanta with zero longitudinal momentum. In addition to the interactions considered in [30], these include a supercurrent-supercurrent interaction mediated by gravitino exchange, as well as membrane-fivebrane and zerobrane-sixbrane interactions (previously considered in [31, 32]). We give explicit matrix expressions for the integrated stress tensor, membrane current and 5-brane current of general matrix theory objects, including the fermionic contributions to these currents. We identify the fermionic contribution to the first moments of the currents, generalizing the previously known result for the spin contribution to graviton angular momentum [20]. We are also able to identify matrix expressions for the fermionic components of the supercurrent as well as a 6-brane current and its first moments.

One motivation for the work described in this paper is the goal of developing a systematic formalism for describing general N-body interactions in matrix theory in terms of structures such as the stress tensor and membrane current which have natural interpretations in supergravity. The nonlinear structure of gravity can be probed by studying 3-body processes in matrix theory. As was mentioned in [27], however, knowledge of the full two-body interaction between objects with nonzero fermionic degrees of freedom is a key step towards carrying out a general calculation of 3-body interactions even between three purely bosonic configurations.

Another application of the results in this paper is to the formulation of matrix theory in a general metric and 3-form background. If the matrix theory conjecture is correct, then the effective action for matrix theory in a wide class of backgrounds can be determined by including extra background blocks in the matrices and integrating out the off-diagonal fields connecting to the background. Armed with a precise knowledge of the form of the matrix theory stress current, membrane and 5-brane currents, we propose a concrete algorithm for constructing the matrix theory action in a general background, and we explicitly give the leading terms for this action in the presence of a weak background field.

It has been proposed that the full set of matrix theory interaction terms which correspond to classical 2-body interactions in gravity should take the form of a nonabelian Born-Infeld theory [33, 34, 35]. Assuming this to be true, our result for the $1/r^7$ term in the effective 2-body potential should be related to a supersymmetric nonabelian Born-Infeld action. We compare our results to previous results on supersymmetric Born-Infeld theory [36, 37] and find that this correspondence seems to hold.

In section 2 we calculate to order $1/r^7$ the complete one-loop matrix theory effective potential for a background corresponding to general widely separated systems. We also calculate the potential to order $1/r^8$ including all terms which are at most quadratic in the fermion backgrounds. As the calculations are somewhat involved, we present only a summary of some salient features in section 2, leaving a more detailed account to appendix A. The results for the effective potential are summarized in section 2.4.

In section 3 we describe the supergravity interactions at orders $1/r^7$ and $1/r^8$ which arise from the exchange of gravitons, gravitinos and 3-form quanta. We show that these interactions correspond precisely with the matrix theory potential computed in section 2 when the proper identification is made of the matrix theory stress tensor, membrane current, 5-brane current, 6-brane current and fermionic components of the supercurrent. The expressions we find for the supergravity currents are summarized in subsection 3.6.

Section 4 contains a discussion of our results, a comparison to other related results in the literature, and some suggestions for how the results of this paper might be applied. In particular, we discuss the connection to higher-order terms in the nonabelian Born-Infeld theory, we briefly discuss the application to the general 3-body problem, we mention possible generalizations to higher-dimensional theories, and we give a concrete suggestion for a formulation of matrix theory in a general background using the matrix forms we have found for the supergravity currents.

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2. Calculation of the effective action

We wish to study the Matrix Theory interaction between an arbitrary pair of isolated systems, whose centers of mass are separated at time t by a distance r which is large compared to the sizes of the two individual systems. In particular, we wish to calculate the matrix theory effective action around a background which is block-diagonal in both bosonic and fermionic matrices.

There are a number of approaches which can be used to perform such a calculation. The simplest approach is to use the quasi-static approximation to calculate the ground state energy of the harmonic oscillator modes corresponding to the off-diagonal degrees of freedom in the theory. This approach was used in [38] to compute the leading term in the interaction potential between a pair of purely bosonic matrix theory systems. Unfortunately, in the presence of fermionic backgrounds this approach cannot be applied in a straightforward fashion. The terms in the tree-level action which are quadratic in the off-diagonal fluctuations include products of bosonic and fermionic fluctuations, complicating the problem significantly. Furthermore, as we shall discuss in more detail, unlike the case of a purely bosonic background, in the case of a completely general background there are non-vanishing terms involving fermions which appear in the effective action below order $1/r^7$. These terms vanish when we enforce the matrix theory equations of motion to calculate the effective potential between two physical systems; the fact that such terms appear already at order $1/r^3$, however, makes the validity of the quasi-static approximation in the general case rather questionable.

Another approach one might consider using to calculate the effective potential for general backgrounds would be to find a supersymmetric completion of the bosonic potential of the form F^4/r^7 computed in [38]. Unfortunately, however, the supersymmetry transformations under which the tree level action is invariant are corrected by subleading terms [18]. Determining precisely how the supersymmetry transformations are modified seems to be just as difficult a problem as doing the full calculation of the effective action.

In this paper we chose to do the calculation using the most conceptually straightforward approach: an explicit summation over all one-loop diagrams in the chosen background. This approach leads to a rather lengthy but finite calculation. In this section we describe the tools used to do the calculation and give brief descriptions of each of the pieces of the calculations. More details are included in appendix A for the reader interested in following the calculations in detail or carrying out analogous calculations themselves. The reader primarily interested in the results may wish to skip from section 2.1 directly to the summary of results for the effective action in section 2.4

2.1 Action and propagators

We begin with the matrix theory action[1]

$$S = -\frac{1}{2R}\int d\tau \mathrm{Tr}\left\{ -D_\tau \mathbf{X}_i D_\tau \mathbf{X}_i + \frac{1}{2}[\mathbf{X}_i, \mathbf{X}_j][\mathbf{X}_i, \mathbf{X}_j] - \left(D_a^{\mathrm{cl}} \mathbf{X}_a^{\mathrm{qu}}\right)^2 + \right.$$
$$\left. + \Theta_\alpha D_\tau \Theta_\alpha - \Theta_\alpha \gamma^i_{\alpha\beta} [\mathbf{X}_i, \Theta_\beta] \right\},$$

with a covariant background-field gauge fixing term

$$D_a^{\mathrm{cl}} \mathbf{X}_a^{\mathrm{qu}} \equiv D_\tau \mathbf{X}_0 + i[B_i, \mathbf{X}_i]$$

plus the corresponding ghost action not written here. We expand the bosonic and fermionic matrices in terms of background and fluctuation degrees of freedom

$$\mathbf{X}_a = \begin{bmatrix} \hat{r}_a + \hat{X}_a + \hat{Z}_a & Y_a \\ Y_a^\dagger & \tilde{r}_a + \tilde{X}_a + \tilde{Z}_a \end{bmatrix}, \qquad \Theta_\alpha = \begin{bmatrix} \hat{\theta}_\alpha + \hat{\eta}_\alpha & \chi_\alpha \\ \chi_\alpha^\dagger & \tilde{\theta}_\alpha + \tilde{\eta}_\alpha \end{bmatrix}, \tag{2.1}$$

where r, X and θ describe background fields and Y, Z, χ and η are the fluctuating fields. The two systems are described respectively by hatted matrices of size $\hat{N} \times \hat{N}$ and tilded matrices of size $\tilde{N} \times \tilde{N}$. We take $\hat{r}_0 = 0$, $\tilde{r}_0 = 0$, while $\hat{r}_i$ and $\tilde{r}_i$ are constants, interpreted as the centers of mass of the two systems at $t = 0$ ($\mathrm{Tr}\,(X_i)$ is taken to vanish at $t = 0$ for both systems).

We would like to compute the effective action obtained by integrating out the off-diagonal fields Y and χ to one loop. For this purpose, we need only keep the terms quadratic in these fields, and we find that the relevant terms in the action (including the term quadratic in off-diagonal ghost fields) are:

$$S_{quad} = -\frac{1}{R}\int d\tau \left[Y_a^\dagger \left((\partial_\tau^2 - r^2 - 2r\cdot K - K^2 + i\dot{K}_0 + 2iK_0\partial_\tau)\delta_{ab} + 2iF_{ab}\right) Y_b + \right.$$
$$+ \chi^\dagger(\partial_\tau - \not{r} - \not{K})\chi + \chi^\dagger \gamma^a L Y_a + Y_a^\dagger L \gamma^a \chi + \tag{2.2}$$
$$\left. + C^*(\partial_\tau^2 - r^2 - 2r\cdot K - K^2 + i\dot{K}_0 + 2iK_0\partial_\tau)C \right].$$

Here, we are treating Y_a and χ_α as $(\hat{N}\tilde{N})$-component vectors acted on by matrices

$$\begin{aligned} K_i &\equiv \hat{X}_i \otimes \mathbb{1}_{\tilde{N}\times\tilde{N}} - \mathbb{1}_{\hat{N}\times\hat{N}} \otimes \tilde{X}_i^T, \\ L_i &\equiv \hat{\theta}_i \otimes \mathbb{1}_{\tilde{N}\times\tilde{N}} - \mathbb{1}_{\hat{N}\times\hat{N}} \otimes \tilde{\theta}_i^T \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} F_{0i} &= \partial_\tau K^i + i[K_0, K^i] \\ F_{ij} &= i[K^i, K^j]. \end{aligned}$$

[1] We use Euclidian conventions, taking $t \to i\tau$, $A \to -i\mathbf{X}_0$. Throughout the work, indices $i, j, k, \ldots$ run from 1 to 9 while indices $a, b, c, \ldots$ run from 0 to 9.

We have defined

$$\gamma^0 = -i$$

(though this does not satisfy the usual anticommutation relations with the other gamma matrices). The only contribution from the quadratic ghost term here will be a set of terms containing purely bosonic fields which will cancel terms from the boson and fermion loops.

For our calculation, we shall treat all terms involving the matrices K, L and F as vertices, using only the terms

$$S_{prop} = -\frac{1}{R}\int d\tau (Y_a^\dagger(\partial_\tau^2 - r^2)Y_a + \chi^\dagger(\partial_\tau - \not{r})\chi)$$

to determine the propagators. These are given simply by:

$$\begin{aligned}\langle Y_a^{kl}(\tau) Y_b^{\dagger\, mn}(\sigma)\rangle &= \delta_{ab}\delta^{lm}\delta^{kn}\int \frac{dk}{2\pi}\frac{e^{ik(\tau-\sigma)}}{k^2+r^2} \\ &\equiv \delta_{ab}\delta^{lm}\delta^{kn}\Delta(\tau-\sigma)\,,\end{aligned}$$

for the bosonic propagator, and

$$\begin{aligned}\langle \chi_\alpha^{kl}(\tau)\chi_\beta^{\dagger\, mn}(\sigma)\rangle &= \delta^{lm}\delta^{kn}\int \frac{dk}{2\pi}\frac{e^{ik(\tau-\sigma)}(\not{r}+ik)_{\alpha\beta}}{k^2+r^2} \\ &\equiv \delta^{lm}\delta^{kn}(\not{r}+\partial_\tau)_{\alpha\beta}\Delta(\tau-\sigma)\,,\end{aligned}$$

for the fermionic propagator. We thus have a fermion-fermion vertex

$$-\chi^\dagger \not{K}\chi\,, \tag{2.4}$$

two mixed vertices

$$\chi^\dagger\gamma^a L Y_a\,, \tag{2.5}$$
$$Y_a^\dagger L\gamma^a\chi \tag{2.6}$$

and boson-boson vertices given by

$$iY_a^\dagger \dot{K}_0 Y_b\,, \tag{2.7}$$
$$2iY_a^\dagger K_0\partial_\tau Y_a\,, \tag{2.8}$$

and

$$-Y_a^\dagger M_{ab} Y_b\,, \tag{2.9}$$

where we define:

$$M_{ab} = 2r\cdot K + K^2 - 2iF_{ab}\,.$$

2.2 Tools

Before proceeding with the calculation, we make a few observations which help to simplify the calculation considerably.

2.2.1 Gauge invariance

First, we may take advantage of the 0+1 dimensional gauge symmetry, noting that by our choice of background field gauge, the effective action we calculate should be invariant under a $0+1$ dimensional gauge transformation. In particular, the result must contain only covariant derivatives, with ∂_τ and K_0 appearing in the combination $\partial_\tau X + i[K_0, X]$. Thus, we could set $K_0 = 0$, calculating terms with all numbers of derivatives and restoring K_0's in the end by replacing derivatives with covariant derivatives. Alternatively, we may restrict the calculation to non-derivative terms, keeping a non-zero K_0, and deduce the derivative terms from the K_0 terms. We will use both approaches, depending on which part of the calculation we are interested in. One approach may also be used as a check of the other.

2.2.2 0+0 dimensional calculation

At this point, we note that the action (2.2) above is closely related to the $0+0$ dimensional action

$$S_{0+0} = -\frac{1}{R}\left(Y_a^\dagger(-r^2\delta_{ab} - M_{ab})Y_b - \chi^\dagger(\not{\eta} + \not{K})\chi + \chi^\dagger\gamma^a L Y_a + Y_a^\dagger L\gamma^a\chi\right) , \qquad (2.10)$$

which arises from the dimensional reduction of the theory to 0 dimensions. This is the quadratic part in fluctuations of the (0+0)-dimensional action which was used by Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT) to conjecture a matrix formulation of IIB string theory [39]. If we restrict to considering terms involving no derivatives or factors of K_0, we find that the $(0+0)$- and $(0+1)$-dimensional calculations are exactly analogous, except that the $(0+0)$-dimensional propagators are simply

$$\langle Y_a^{kl}(\tau) Y_b^{\dagger\, mn}(\sigma)\rangle = \delta_{ab}\delta^{lm}\delta^{kn}\frac{1}{r^2},$$

and

$$\langle \chi_\alpha^{kl}(\tau)\chi_\beta^{\dagger\, mn}(\sigma)\rangle = \delta^{lm}\delta^{kn}\frac{\not{\eta}_{\alpha\beta}}{r^2} .$$

The form of the leading $1/r^8$ term in the bosonic 2-body potential in the $(0+0)$-dimensional theory was computed by IKKT in [39] and is the same up to a constant as the $1/r^7$ 2-body term calculated for matrix theory in [38]. As we shall see below, the answer from the simpler $0+0$ dimensional calculation will often aid us in computing the full $0+1$ dimensional result.

2.2.3 $r \cdot K$ and $\not{r}$ terms

A further simplification arises from the fact that in the action (2.2), r_i and K_i appear only in the combination $(r_i + K_i)$ (a trivial consequence of their definitions). Thus, a transformation

$$r_i \to r_i + \Lambda_i \tag{2.11}$$

is exactly equivalent to a transformation

$$K_i \to K_i + \Lambda_i \,. \tag{2.12}$$

As described in detail in the body of the calculation, this property will allow us to ignore all terms containing $r \cdot K$ and certain terms with an r^i coupled to a gamma matrix during the calculation, and to deduce these at the end using a transformation (2.12) on the remaining terms.

2.2.4 D=10 notation

It turns out that a certain subset of terms in the effective action at each order are the dimensional reduction of $D = 10$ Lorentz and gauge invariant terms, while many of the remaining terms are simply related to these by insertions of operators such as $\vec{K}^2$, $r \cdot K$ and $(D_0)^2$ which are not invariant under the $D = 10$ symmetries but which do have the $D = 0+1$ dimensional gauge invariance and SO(9) rotational invariance of the Matrix Theory action.

Thus, we will find that the fermionic terms in the effective action may often be written most naturally and concisely using $D = 10$ gamma matrices defined by

$$\Gamma^i = \begin{bmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{bmatrix}, \qquad \Gamma^0 = \begin{bmatrix} 0 & \gamma^0 \\ -\gamma^0 & 0 \end{bmatrix} \tag{2.13}$$

and a 32 component spinor L whose first sixteen components are taken to vanish and whose remaining components are the 16 component spinor considered previously. Note that these gamma matrices correctly satisfy the relations defining the $D = 10$ Clifford algebra,

$$\{\Gamma^a, \Gamma^b\} = 2\delta_{ab} \,.$$

Given that the original action has a $D = 10$ covariant form, one might wonder why not all terms in the one loop effective action may be written in this way. However, in computing the one loop potential, we integrate over only a single momentum rather than a ten-component momentum vector. Further, in writing our result, we make an expansion in the transverse seperation r. Neither of these respect the ten dimensional structure that we started with, so we should not expect the full one loop effective action to be the dimensional reduction of the $D = 10$ result.

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2.3 Summary of the calculation

We divide the calculation up by the number of fermionic fields appearing in the result. The leading term with $2n$ fermions comes from choosing n each of the vertices (2.5) and (2.6). Such a term will have n bosonic and n fermionic propagators, and thus appear at order $(1/r)^{3n-1}$. Thus, up to order $1/r^7$, where we expect the leading-order interactions, we will have terms with zero, two, and four fermions.

2.3.1 Bosonic terms

The purely bosonic terms at one loop were previously calculated for a general background in [38], using the quasistatic approximation in which the background fields (and their derivatives) appearing in the action (2.2) were assumed to be time-independent. Since we are no longer restricting to this approximation, it is interesting to check whether any higher derivative terms appear up to $1/r^7$. In our calculation, these extra derivative terms come from keeping higher order terms in the Taylor expansions of the background fields about a particular time, rather than assuming that the background fields are time independent.

Calculating with $K_0 = 0$, the non-vanishing vertices containing only bosonic background fields are (2.4), (2.9), and the ghost vertex. The only diagrams we may build with these are loops with a single one of these vertex types repeated arbitrarily many times. These three types of contributions are calculated explicitly in appendix A.1, and it is found that all terms with explicit derivatives cancel up to and including order $1/r^7$, leaving exactly the action

$$\Gamma^{\text{bos}}_{1/r^7} = \int d\tau \frac{15}{16r^7} \text{STr}\left(F_{ab}F_{bc}F_{cd}F_{da} - \frac{1}{4}F_{ab}F_{ab}F_{cd}F_{cd}\right) \tag{2.14}$$

calculated in [38]. Here, STr denotes a symmetrized trace in which we average over all possible orderings of the matrices in the trace (treating any commutators such as F as a unit).

Thus, to this order the result of our full calculation agrees with the quasistatic calculation. However, as we shall see below, we expect the appearance of higher-derivative terms at order $1/r^9$ which would not appear in a quasistatic approximation. These are proportional to the average of all possible insertions of D_0^2 (covariant derivative squared) into the terms (2.14) of the $1/r^7$ action. Interestingly, the cancellation up to order $1/r^7$ is off shell, that is, it does not require use of the equations of motion. We will see that this is not the case for terms with fermionic fields.

We will also be interested in terms of order $1/r^8$. At this order, we find two purely bosonic contributions. The first, studied in [40, 30], is proportional to an insertion of $r \cdot K$ into the $1/r^7$ action,

$$-\frac{105}{16}\text{STr}\left(F_{ab}F_{bc}F_{cd}F_{da}(r\cdot K) - \frac{1}{4}F_{ab}F_{ab}F_{cd}F_{cd}(r\cdot K)\right).$$

The second contribution contains a nine index totally antisymmetric tensor coming from the trace of nine gamma matrices, and is given by

$$\frac{35r_s}{256r^9}\mathrm{STr}\,(F_{ij}F_{kl}F_{mn}\dot{K}_pK_q)\epsilon^{ijklmnpqs}\,.$$

We will discuss the physical interpretation of these two terms below.

2.3.2 Two-fermion terms

We now move on to terms in the effective action with two factors of the background fermion fields L. These terms arise from loops with a single insertion of each of the two boson-fermion vertices (2.5) and (2.6) to give two factors of L. In addition, we may have an arbitrary number of boson-boson vertices and fermion-fermion vertices.

The details of the calculation, performed using a non-vanishing K_0 and ignoring derivative terms, are found in appendix A.2. In contrast to the case of the purely bosonic terms, we find that the off-shell action does not vanish below $1/r^7$, and in fact there are non-vanishing contributions to the effective action starting at $1/r^3$. However, if we are interested in the effective potential between two physical systems, we must require that the background fields satisfy the Matrix Theory equations of motion, and in this case, the non-vanishing contributions begin at order $1/r^7$, in agreement with supergravity.

We now give our result for the two-fermion terms in the one loop matrix theory effective action to order $1/r^7$. As mentioned above, we have a subset of terms which are the dimensional reduction of a $D=10$ Lorentz and gauge invariant action. These are given in $D=10$ notation (omitting the leading $1/r^n$) by

$$\begin{aligned}
\Gamma^{\mathrm{cov}}_{1/r^3} &= \mathrm{Tr}\,(\bar{L}\not{D}L)\\
\Gamma^{\mathrm{cov}}_{1/r^5} &= \frac{3}{8}i\,\mathrm{Tr}\,(\bar{L}\,\Gamma^b\,D_aF_{ab}\,L)+\\
&\quad+\frac{3}{16}i\,\mathrm{Tr}\,(\bar{L}\,\Gamma^{[ab]}\,F_{ab}\,\not{D}L)-\frac{3}{16}i\,\mathrm{Tr}\,(\bar{L}\,\Gamma^{[ab]}\,\not{D}L\,F_{ab})\\
\Gamma^{\mathrm{cov}}_{1/r^7} &= -\frac{5}{64}\mathrm{Tr}\,(\bar{L}\,\Gamma^e\Gamma^a\Gamma^b\Gamma^d\,F_{ab}\,\not{D}L\,F_{de})-\\
&\quad-\frac{5}{16}\mathrm{Tr}\,(\bar{L}\,\Gamma^a\Gamma^c\,F_{ab}\,F_{bc}\,\not{D}L)-\frac{5}{16}\mathrm{Tr}\,(\bar{L}\,\Gamma^c\Gamma^a\,\not{D}L\,F_{ab}\,F_{bc})+\\
&\quad+\frac{5}{32}\mathrm{Tr}\,(\bar{L}\,\Gamma^d\Gamma^b\Gamma^c\,D_aF_{ab}\,L\,F_{cd})-\\
&\quad-\frac{5}{16}\mathrm{Tr}\,(\bar{L}\,\Gamma^c\,D_aF_{ab}\,L\,F_{cb})-\frac{5}{16}\mathrm{Tr}\,(\bar{L}\,\Gamma^c\,F_{cb}\,D_aF_{ab}\,L)+\\
&\quad+\frac{15}{16}\mathrm{STr}\,(\bar{L}\,\Gamma^b\Gamma^c\Gamma^d\,F_{ab}\,F_{cd}\,D_aL)
\end{aligned}\qquad(2.15)$$

All other terms up to order $1/r^7$ come from insertions of $r\cdot K$, $\vec{K}^2$, or D_0^2 into the covariant terms. To write these, we define an operation $\mathrm{Sym}\,(\mathrm{Tr}\,(A_1\ldots A_n);B_1,\ldots,B_n)$

to be the average of all possible different insertions of the operators B_i between elements of the trace. For example,

$$\begin{aligned}\mathrm{Sym}\,(\mathrm{Tr}\,(\bar{L}\;\not{D}L);\vec{K}^2,D_0^2) = \frac{1}{6}\Big[&\mathrm{Tr}\,(D_0^2\bar{L}\;\vec{K}^2\;\not{D}L)+\mathrm{Tr}\,(\bar{L}\;D_0^2\vec{K}^2\;\not{D}L)+\\ &+\mathrm{Tr}\,(\bar{L}\;\vec{K}^2\;D_0^2\not{D}L)+\mathrm{Tr}\,(D_0^2\bar{L}\;\not{D}L\;\vec{K}^2)+\\ &+\mathrm{Tr}\,(\bar{L}\;D_0^2\not{D}L\;\vec{K}^2)+\mathrm{Tr}\,(\bar{L}\;\not{D}L\;D_0^2\vec{K}^2)\Big]\end{aligned}$$

Note that any commutators in the trace are treated as single element. We can now write the full set of two-fermion terms in the $D = 0+1$ off-shell action up to order $1/r^7$ as (again omitting factors of $1/r^n$)

$$\begin{aligned}\Gamma_{1/r^3} &= \Gamma^{\mathrm{cov}}_{1/r^3}\\ \Gamma_{1/r^4} &= -3\;\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};r\cdot K)\\ \Gamma_{1/r^5} &= \Gamma^{\mathrm{cov}}_{1/r^5}+\frac{1}{4}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};D_0^2)-\frac{3}{2}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};\vec{K}^2)+\frac{15}{2}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};r\cdot K,r\cdot K)\\ \Gamma_{1/r^6} &= -5\;\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^5};r\cdot K)-\frac{15}{8}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};D_0^2,r\cdot K)+\\ &\quad+\frac{15}{2}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};\vec{K}^2,r\cdot K)-\frac{35}{2}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};r\cdot K,r\cdot K,r\cdot K)\\ \Gamma_{1/r^7} &= \Gamma^{\mathrm{cov}}_{1/r^7}+\frac{5}{8}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^5};D_0^2)-\frac{5}{2}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^5};\vec{K}^2)+\frac{35}{2}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^5};r\cdot K,r\cdot K)+\\ &\quad+\frac{15}{8}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};\vec{K}^2,\vec{K}^2)-\frac{15}{16}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};D_0^2,\vec{K}^2)+\frac{1}{16}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};D_0^2,D_0^2)-\\ &\quad-\frac{105}{4}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};\vec{K}^2,r\cdot K,r\cdot K)+\frac{105}{48}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};D_0^2,r\cdot K,r\cdot K)+\\ &\quad+\frac{315}{8}\mathrm{Sym}\,(\Gamma^{\mathrm{cov}}_{1/r^3};r\cdot K,r\cdot K,r\cdot K,r\cdot K)\,. \end{aligned}\tag{2.16}$$

We will see that things simplify greatly when we restrict to a background that satisfies the matrix theory equations of motion. In our $D = 10$ notation, these equations of motion read

$$\not{D}L = 0 \tag{2.17}$$

and

$$D_a F_{ab} = i\bar{L}\Gamma^b L \tag{2.18}$$

Examining the covariant terms above, we see that apart from the final term in the $1/r^7$ action, all terms listed contain either $\not{D}L$ or $D_a F_{ab}$, and so these terms either cancel directly or may be cancelled by the four-fermion terms which we will soon calculate.

In the applications below, we will also be interested in certain terms at $1/r^8$. We find that apart from terms containing $r\cdot K$, the two fermion terms at $1/r^8$ are given by the $D = 10$ gauge invariant expression

$$\Gamma_{1/r^8} = -\frac{35}{256 r^9} r_g \mathrm{STr}\left(\bar{L}\, F_{ab} F_{cd} F_{ef} \Gamma^{[abcdefg]} L\right) - \frac{105}{32 r^9} r_e \mathrm{STr}\left(\bar{L}\, F_{ab} F_{bc} F_{cd} \Gamma^{[ade]} L\right) - \frac{105}{128 r^9} r_e \mathrm{STr}\left(\bar{L}\, F_{ab} F_{ab} F_{cd} \Gamma^{[cde]} L\right). \qquad (2.19)$$

Again, we'll see the physical interpretation of these terms below.

2.3.3 Four-fermion terms

The remaining terms in the one-loop Matrix theory effective action up to order $1/r^7$ contain four fermions. These arise from a loop with two each of the vertices (2.5) and (2.6) plus insertions of the boson-boson vertices and the fermion-fermion vertex.

The calculation of four-fermion terms, performed setting K_0 to 0 and calculating derivative terms explicitly, appears in appendix A.3. The calculation relies heavily on the use of Fierz identities for 16×16 symmetric gamma matrices, and a discussion of these as they apply to non-abelian four-fermion terms is given in appendix B.

We find first a set of terms (A.18), (A.25), (A.49) listed in the appendix which are obtained simply by replacing $D_a F_{ab}$ with $D_a F_{ab} - i\bar{L}\Gamma^b L$ each time it appears in the two-fermion terms. These terms serve to cancel all two-fermion terms containing $D_a F_{ab}$ in the on-shell effective action.

The remaining four-fermion terms which do not contain $\not{D}L$ all appear at $1/r^7$, so we indeed have complete cancellation of all terms below $1/r^7$ in the on-shell effective potential, in agreement with supergravity. These remaining four-fermion terms are given by

$$\begin{aligned}
\Gamma^{4L}_{1/r^7} = & \frac{5}{32}\Big(\mathrm{Tr}\,(\bar{L}\Gamma^a D_b L \bar{L}\Gamma^a D_b L) + \mathrm{Tr}\,(D_b\bar{L}\Gamma^a L D_b \bar{L}\Gamma^a L) + 2\mathrm{Tr}\,(\bar{L}\Gamma^a L D_b \bar{L}\Gamma^a D_b L)\Big) + \\
& + \frac{5i}{32}\Big(\mathrm{Tr}\,(\bar{L}\Gamma^c F_{ab} L \bar{L}\Gamma^{[cab]} L) + \mathrm{Tr}\,(\bar{L}\Gamma^{[cab]} L F_{ab} \bar{L}\Gamma^c L) + \mathrm{Tr}\,(\bar{L}\Gamma^c \bar{L} F_{ab} L \Gamma^{[cab]} L)\Big) + \\
& + \frac{5}{128}\Big(\mathrm{Tr}\,(L\gamma^{[ki]}[\dot{K}_i, L]\; L\gamma^k L) + \mathrm{Tr}\,(L\gamma^{[ki]} L [\dot{K}_i, L]\gamma^k L)\Big) - \\
& - \frac{5}{128}\Big(\mathrm{Tr}\,(L\gamma^{[kl]}\dot{L}\; \dot{L}\gamma^{[lk]} L) + 2\mathrm{Tr}\,(L\gamma^k \dot{L}\; \dot{L}\gamma^k L) + 6\mathrm{Tr}\,(L\dot{L}\; \dot{L}L)\Big) + \\
& + \frac{5}{256}\Big(\mathrm{Tr}\,(L\gamma^{[kli]} K_i \dot{L}\; L\gamma^{[lk]} L) + 6\mathrm{Tr}\,(L\not{K}\dot{L}\; LL) + 2\mathrm{Tr}\,(LK_i \dot{L}\; L\gamma^i L) - \\
& \qquad - \mathrm{Tr}\,(\dot{L}\gamma^{[kli]} K_i L\; L\gamma^{[lk]} L) - 6\mathrm{Tr}\,(\dot{L}\not{K}L\; LL) - 2\mathrm{Tr}\,(\dot{L} K_i L\; L\gamma^i L)\Big) + \\
& + \frac{15}{256}\Big(\mathrm{Tr}\,(L\gamma^{[kli]} K_i L\; \dot{L}\gamma^{[lk]} L) + 6\mathrm{Tr}\,(L\not{K}L\; \dot{L}L) + 2\mathrm{Tr}\,(LK_i L\; \dot{L}\gamma^i L) - \\
& \qquad - \mathrm{Tr}\,(L\gamma^{[kli]} K_i L\; L\gamma^{[lk]} \dot{L}) - 6\mathrm{Tr}\,(L\not{K}L\; L\dot{L}) - 2\mathrm{Tr}\,(LK_i L\; L\gamma^i \dot{L})\Big) + \\
& + \frac{5}{64}\Big(\mathrm{Tr}\,(L\gamma^{[kli]} K_i L\; L\gamma^{[lkj]} K_j L) + 2\mathrm{Tr}\,(L\gamma^{[ki]} K_i L\; L\gamma^{[kj]} K_j L) + \\
& \qquad + 6\mathrm{Tr}\,(L\gamma^i K_i L\; L\gamma^j K_j L) + 2\mathrm{Tr}\,(L\gamma^j K_i L\; L\gamma^i K_j L) - \\
& \qquad - 2\mathrm{Tr}\,(L\gamma^i K_j L\; L\gamma^i K_j L) + 2\mathrm{Tr}\,(LK_i L\; LK_i L)\Big). \qquad (2.20)
\end{aligned}$$

Here, we note that some of the terms have a $D = 10$ form, while others do not seem to combine into $D = 10$ covariant expressions. In particular, it appears that there is no way to rewrite the above expression in a form with all K's appearing in commutators, even taking into account all possible Fierz identities.

2.4 Summary of results for effective potential

We have now calculated the complete one-loop matrix theory effective action to order $1/r^7$ for a background corresponding to two widely separated systems. The off-shell action is given by expressions (2.15), (2.16), (2.20), (A.18), (A.25) and (A.49) and has non-vanishing contributions beginning at order $1/r^3$. However, as we saw above, enforcing the Matrix theory equations of motion (2.17) and (2.18) we find complete cancellation of all terms at orders less than $1/r^7$, in agreement with supergravity. The on-shell effective action (which we may interpret as the negative of an effective potential for the system) is given by

$$\begin{aligned}\Gamma_{1/r^7} = & \frac{15}{16r^7}\mathrm{STr}\left(F_{ab}F_{bc}F_{cd}F_{da} - \frac{1}{4}F_{ab}F_{ab}F_{cd}F_{cd}\right) + \\ & + \frac{15}{16r^7}\mathrm{STr}\,(\bar{L}\,\Gamma^b\Gamma^c\Gamma^d\; F_{ab}\,F_{cd}\,D_a L) + \Gamma^{4L} \qquad (2.21)\end{aligned}$$

where Γ^{4L} is given in (2.20) (replacing, if we like, factors of $\dot{L}$ by $[\not{K}, L]$). At $1/r^8$, we have terms proportional to the insertion of $r \cdot K$ into the symmetrized trace of the $1/r^7$ terms plus additional terms. The full expression to quadratic order in in the fermion backgrounds is given by

$$\begin{aligned}\Gamma_{1/r^8} = & -7\,\mathrm{Sym}\,(\Gamma_{1/r^7}; r \cdot K) + \frac{35r_s}{256r^9}\mathrm{STr}\,(F_{ij}F_{kl}F_{mn}\dot{K}_p K_q)\epsilon^{ijklmnpqs} - \\ & - \frac{35}{256r^9}r_g\mathrm{STr}\,(\bar{L}\,F_{ab}F_{cd}F_{ef}\Gamma^{[abcdefg]}L) - \frac{105}{32r^9}r_e\mathrm{STr}\,(\bar{L}\,F_{ab}F_{bc}F_{cd}\Gamma^{[ade]}L) - \\ & - \frac{105}{128r^9}r_e\mathrm{STr}\,(\bar{L}\,F_{ab}F_{ab}F_{cd}\Gamma^{[cde]}L) \qquad (2.22)\end{aligned}$$

There are also four-fermion and six-fermion terms which contribute to Γ_{1/r^8} which we have not calculated. In the next section, we will compare the terms in (2.21) and (2.22) to leading-order supergravity interactions, and will find a complete interpretation for all of these terms in terms of physical quantities familiar from supergravity.

3. Comparison to supergravity

In order to compare our Matrix theory result with supergravity, we would like to understand from a supergravity point of view the interactions that exist between two arbitrary widely separated systems at leading orders in the inverse separation distance. The leading terms in the long-range effective potential between two separated systems are described in supergravity by the linearized theory. The terms in the linearized theory arising from graviton and 3-form exchange between arbitrary bosonic systems were analyzed in [30] and shown to agree with matrix theory for processes with no longitudinal momentum transfer. In this section we generalize that discussion to include fermionic backgrounds. This gives rise to additional contributions from the fermion backgrounds to the gravitational, "electric" and "magnetic" interactions mediated by the graviton and 3-form quantum, as well as new interactions arising from gravitino exchange. Extending the analysis out to order $1/r^8$ we also find new interactions in the bosonic sector describing "dyonic" membrane-5-brane and 0-brane-6-brane interactions.

All the interactions in the linearized theory can be written in a current-current form corresponding to an instantaneous potential in light-front time between the two separated systems proportional to the product of moments of the stress tensor, membrane current, 5-brane current, 6-brane current and fermionic components of a supercurrent. To identify the matrix theory and supergravity interaction potentials the only structure needed is the detailed form of these currents in matrix theory language, expressed as traces of products of the backgrounds for the two systems. A synopsis of our results for these currents is given in section 3.6.

3.1 Linearized supergravity interactions

In this subsection we discuss the general structure of the linearized supergravity interactions we expect to reproduce in matrix theory. The detailed forms of the specific interactions are discussed and compared to matrix theory in the following subsections. The propagating fields of eleven dimensional supergravity are a graviton, a three form gauge field, and a gravitino. The classical linearized supergravity theory arises from considering all tree-level processes in which a single quantum is exchanged between two classical sources. For an eleven-dimensional spacetime with one compact direction, the propagators for all the fields go like $1/r^7$, so at orders below $1/r^{14}$ all classical supergravity interactions can be described by the linearized theory.

Arbitrary sources can be coupled linearly to the supergravity fields by adding extra terms to the action of the form

$$S = S_{\mathrm{SUGRA}}(h, A, \psi) + \int d^{11}x \Big(h_{IJ} T^{IJ} + A_{IJK} J^{IJK} + \\ + A^D_{IJKLMN} M^{IJKLMN} + \bar{\psi}_{\alpha I} S^I_\alpha + \bar{S}^I_\alpha \psi_{\alpha I} \Big) , \tag{3.1}$$

where $T^{IJ}, J^{IJK}, M^{IJKLMN}$ and S^I_α are the stress-energy tensor, membrane current, 5-brane current and fermionic supercurrent components of the source. The field A^D

has a 7-form field strength dual[2] to that of the 3-form field A

$$F^D = dA^D = {}^*F = {}^*(dA)\,. \tag{3.2}$$

It is difficult to formulate a consistent quantum theory which contains both A and A^D and which couples both to membranes and 5-branes since (3.2) is difficult to impose at the quantum level. We are only interested in the classical theory here, however, so we may impose (3.2) as a classical condition. This still leads to some complications since there is not always a single-valued solution for A^D of (3.2) for a given field A. We discuss this issue further in (3.4).

The classical interactions in the linearized theory can be determined by simply taking the quadratic part of the supergravity action and solving explicitly for the propagating fields in the presence of the given background. If we have a background containing two well-separated objects, we may treat one object as a source, solve for the fields produced by this source, and use (3.1) to find the effective action of the second object which is treated as a probe in the fields produced by the first object.

Another approach to explicitly describing the linearized theory, which is slightly more transparent in the light-front formalism, is to follow the standard field theory prescription for writing the interactions in terms of the propagators of the gravity fields. Keeping only the quadratic terms in the supergravity action, we have interactions arising from graviton, 3-form and gravitino exchange. To study interactions between two isolated systems, we assume that each of the sources in (3.1) may be decomposed into the sum of two terms whose supports are separated by some large distance, for example $T^{IJ} = \hat{T}^{IJ} + \tilde{T}^{IJ}$. The leading-order interactions are then given by all diagrams of the form

$$\hat{\times}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\tilde{\times} \tag{3.3}$$

coupling a hatted source to a tilded source. The following non-vanishing terms appear in the effective potential (up to overall coefficients):

$$V_{\text{gravity}} = \int d^{11}x \int d^{11}y\; \hat{T}^{IJ}(x) \langle h_{IJ}(x) h_{KL}(y) \rangle \tilde{T}^{KL}(y) \tag{3.4}$$

$$V_{\text{electric}} = \int d^{11}x \int d^{11}y\; \hat{J}^{IJK}(x) \langle A_{IJK}(x) A_{LMN}(y) \rangle \tilde{J}^{LMN}(y) \tag{3.5}$$

$$V_{\text{magnetic}} = \int d^{11}x \int d^{11}y\; \hat{M}^{IJKLMN}(x) \langle A^D_{IJKLMN}(x) A^D_{PQRSTU}(y) \rangle \tilde{M}^{PQRSTU}(y) \tag{3.6}$$

$$\begin{aligned} V_{\text{super}} = {} & \int d^{11}x \int d^{11}y\; \bar{\hat{S}}{}^I_\alpha(x) \langle \psi_{\alpha I}(x) \bar{\psi}_{\beta J}(y) \rangle \tilde{S}^J_\beta(y) - \\ & + \int d^{11}x \int d^{11}y\; \bar{\tilde{S}}{}^I_\alpha(x) \langle \psi_{\alpha I}(x) \bar{\psi}_{\beta J}(y) \rangle \hat{S}^J_\beta(y)\,. \end{aligned} \tag{3.7}$$

[2] The true duality relation is somewhat more complicated, however we ignore the additional terms in the linearized theory since they are products of more than one field.

In addition, there are membrane-fivebrane interactions of the form $\hat{J}\tilde{M}$ and $\hat{M}\tilde{J}$ proportional to a propagator $\langle A(x)A^D(y)\rangle$. These terms, however, cannot be completely described by a potential. These terms are discussed separately in section (3.4) below.

To compare with matrix theory, we must take supergravity on a spacetime with a lightlike direction compactified, and also consider only processes involving the exchange of quanta with zero longitudinal momentum. The appropriate Green's function and propagators for the bosonic fields are discussed and calculated in [30]. Denoting the light-front time coordinate by x^+, the compact direction by x^-, and the remaining spatial directions by $\vec{x}$, the appropriate propagators are all proportional to

$$\delta(x^+ - y^+)\frac{1}{|\vec{x}-\vec{y}|^7}\,.$$

Note that this is independent of x^-, a consequence of the fact that we have restricted to the exchange of quanta with zero momentum in x^-, which therefore have wavefunctions spread evenly over this direction. Also notable is the fact that such a propagator gives rise to interactions which are instantaneous with respect to light-front time (see, for example [41]).

We now show schematically how we can relate these supergravity interactions to the matrix theory potential we have calculated. Examining the interactions (3.4)–(3.7) above, we see that all have the form

$$\int d^{11}x \int d^{11}y\; \hat{C}(x)\langle\phi(x)\phi(y)\rangle\tilde{C}(y)\,, \tag{3.8}$$

where C is a general source and ϕ is a propagating field. Note that in general, a gauge-fixing choice may be necessary to explicitly calculate the propagator and to determine how the components of the tensor currents are contracted. Since the propagator is independent of x^- and y^-, the integrals over these variables act only on the background currents. We may rewrite the interaction as an explicit series in $1/r$ by replacing the currents $C(x)$ by equivalent distributions

$$\begin{aligned}\int dx^- C(x^+,x^-,\vec{x}) = {} & C(x^+)\delta(\vec{x}-\vec{x}_0) - C^{(i)}(x^+)\partial_i\delta(\vec{x}-\vec{x}_0) + \\ & +\frac{1}{2}C^{(ij)}(x^+)\partial_i\partial_j\delta(\vec{x}-\vec{x}_0) + \cdots\,,\end{aligned} \tag{3.9}$$

where we define the spatial moments of the current $C(x)$ about the point $\vec{x}_0$ by

$$C^{(i_1\ldots i_n)}(x^+) = \int dx^- d\vec{x}\; C(x^+,x^-,\vec{x}_0+\vec{x})\; x^{i_1}\cdots x^{i_n}\,.$$

We choose the points $\vec{x}_0$ and $\vec{y}_0$ to be the centres of mass of the two objects at some initial time $t = x^+ = 0$, so that $\vec{x}_0 - \vec{y}_0$ may be identified with the vector $\vec{r}$ defined in our matrix theory background. With these definitions, the supergravity interaction (3.8) may be rewritten as (up to an overall proportionality constant)

$$\int d\tau \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}\hat{C}^{(i_1\ldots i_k)}\tilde{C}^{(j_{k+1}\ldots j_n)}\partial_{i_1}\cdots\partial_{j_n}\left(\frac{1}{r^7}\right) =$$

$$= \int d\tau \left\{\frac{1}{r^7}\hat{C}\tilde{C} + 7\frac{r^i}{r^9}\left(\hat{C}^{(i)}\tilde{C}+\hat{C}\tilde{C}^{(i)}\right)+\cdots\right\} \quad (3.10)$$

Hence the supergravity interactions we are interested in may be written order by order in $1/r$ with terms at each order coupling some moment of the current for the hatted system with a moment of the current for the tilded system.

Let us compare this discussion to the general form of the corresponding potential in matrix theory. In Matrix theory, recalling the definitions (2.3), we see that each term in the one-loop matrix theory effective action is a single trace of a product of matrices which are the tensor product of an $\hat{N}\times\hat{N}$ matrix and a $\tilde{N}\times\tilde{N}$ matrix. Thus, the trace in a given term splits into the product of two traces as

$$\mathrm{Tr}\left((\hat{A}_1\otimes\tilde{A}_1)\cdots(\hat{A}_n\otimes\tilde{A}_n)\right) = \mathrm{Tr}\,(\hat{A}_1\cdots\hat{A}_n)\mathrm{Tr}\,(\tilde{A}_1\cdots\tilde{A}_n)\,. \quad (3.11)$$

Exactly as in the supergravity expression (3.10), the one-loop matrix theory action may be written as a series in $1/r$ with each term coupling a single hatted quantity to a tilded quantity. This suggests that for each moment of a given supergravity current, we may identify a corresponding matrix theory quantity given by a single trace of matrix theory variables.

We will now examine the explicit forms of the supergravity and matrix theory potentials and show that this correspondence is precise, identifying explicit quantities corresponding to moments of the supergravity currents and showing that the supergravity interactions are exactly reproduced in the one loop matrix theory action.

3.2 Gravitational and 3-form interactions at order $1/r^7$

We first consider the interactions (3.4), (3.5), and (3.6) mediated by the exchange of bosonic particles. In [30] these terms were analyzed for purely bosonic backgrounds. It was shown that the leading contribution to the long-range potential of both linearized supergravity and matrix theory could be expressed in the form

$$V_{\mathrm{sugra}} = V_{\mathrm{gravity}} + V_{\mathrm{electric}} + V_{\mathrm{magnetic}} \quad (3.12)$$

$$V_{\mathrm{gravity}} = -\frac{15R^2}{4r^7}\left(\hat{T}^{IJ}\tilde{T}_{IJ} - \frac{1}{9}\hat{T}^I{}_I\tilde{T}^J{}_J\right) \quad (3.13)$$

$$V_{\mathrm{electric}} = -\frac{45R^2}{r^7}\hat{J}^{IJK}\tilde{J}_{IJK} \quad (3.14)$$

$$V_{\mathrm{magnetic}} = -\frac{3R^2}{2r^7}\hat{M}^{IJKLMN}\tilde{M}_{IJKLMN}\,, \quad (3.15)$$

where in the supergravity theory the tensors T^{IJ}, J^{IJK} and M^{IJKLMN} are identified with the integrated stress tensor, membrane current and 5-brane current of the two separated systems. The identification with matrix theory was made by identifying matrix forms for each of these tensors written as traces of functions of the bosonic fields describing the background.

In the more general situation we are considering here with background fermions, we can perform a similar analysis. We can decompose the leading terms in the matrix theory effective potential (2.21) in terms of a sum of products of traces of the bosonic and fermionic degrees of freedom of the two systems as in (3.11), where we only retain terms with bosonic indices contracted between the hatted and tilde'd variables. We find that in general this part of the effective potential (2.21) can be expressed in the form (3.12), where in the presence of fermionic backgrounds the matrix theory expressions for the integrated currents are defined by (returning to a Minkowski formalism):

$$
\begin{aligned}
T^{++} &= \frac{1}{R}\mathrm{STr}\ (\mathbb{1}) \\
T^{+i} &= \frac{1}{R}\mathrm{STr}\ \left(\dot{X}_i\right) \\
T^{+-} &= \frac{1}{R}\mathrm{STr}\ \left(\frac{1}{2}\dot{X}_i\dot{X}_i + \frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\theta\gamma^i[X^i,\theta]\right) \\
T^{ij} &= \frac{1}{R}\mathrm{STr}\ \left(\dot{X}_i\dot{X}_j + F_{ik}F_{kj} - \frac{1}{4}\theta\gamma^i[X_j,\theta] - \frac{1}{4}\theta\gamma^j[X_i,\theta]\right) \\
T^{-i} &= \frac{1}{R}\mathrm{STr}\ \left(\frac{1}{2}\dot{X}_i\dot{X}_j\dot{X}_j + \frac{1}{4}\dot{X}_iF_{jk}F_{jk} + F_{ij}F_{jk}\dot{X}_k\right) - \\
&\quad -\frac{1}{4R}\mathrm{STr}\ \left(\theta_\alpha\dot{X}_k[X_m,\theta_\beta]\right)\{\gamma^k\delta_{im} + \gamma^i\delta_{mk} - 2\gamma^m\delta_{ki}\}_{\alpha\beta} - \\
&\quad -\frac{1}{8R}\mathrm{STr}\ (\theta_\alpha F_{kl}[X_m,\theta_\beta])\{\gamma^{[iklm]} + 2\gamma^{[lm]}\delta_{ki} + 4\delta_{ki}\delta_{lm}\}_{\alpha\beta} + \\
&\quad +\frac{i}{8R}\mathrm{Tr}\,(\theta\gamma^{[ki]}\theta\ \theta\gamma^k\theta) \\
T_f^{--} &= \frac{1}{4R}\mathrm{STr}\ \left(F_{ab}F_{bc}F_{cd}F_{da} - \frac{1}{4}F_{ab}F_{ab}F_{cd}F_{cd} + \theta\Gamma^a\Gamma^b\Gamma^c F_{ab}F_{cd}D_a\theta + \mathcal{O}(\theta^4)\right) \\
J^{+ij} &= \frac{1}{6R}\mathrm{STr}\ (F_{ij}) \\
J^{+-i} &= \frac{1}{6R}\mathrm{STr}\ \left(F_{ij}\dot{X}_j - \frac{1}{2}\theta[X_i,\theta] + \frac{1}{4}\theta\gamma^{[ki]}[X_k,\theta]\right) \\
J^{ijk} &= \frac{1}{6R}\mathrm{STr}\ \left(\dot{X}_iF_{jk} + \dot{X}_jF_{ki} + \dot{X}_kF_{ij} - \frac{1}{4}\theta\gamma^{[ijkl]}[X_l,\theta]\right)
\end{aligned}
$$

$$
\begin{aligned}
J^{-ij} &= \frac{1}{6R}\mathrm{STr}\,\Big(\dot{X}_i\dot{X}_kF_{kj} - \dot{X}_j\dot{X}_kF_{ki} - \frac{1}{2}\dot{X}_k\dot{X}_kF_{ij} + \frac{1}{4}F_{ij}F_{kl}F_{kl} + F_{ik}F_{kl}F_{lj}\Big) + \\
&\quad + \frac{1}{24R}\mathrm{STr}\,\Big(\theta_\alpha\dot{X}_k[X_m,\theta_\beta]\Big)\Big\{\gamma^{[kijm]} + \gamma^{[jm]}\delta_{ki} - \gamma^{[im]}\delta_{kj} + \\
&\qquad\qquad + 2\delta_{jm}\delta_{ki} - 2\delta_{im}\delta_{kj}\Big\}_{\alpha\beta} + \\
&\quad + \frac{1}{8}\mathrm{STr}\,(\theta_\alpha F_{kl}[X_m,\theta_\beta])\Big\{\gamma^{[jkl]}\delta_{mi} - \gamma^{[ikl]}\delta_{mj} + 2\gamma^{[lij]}\delta_{km} + 2\gamma^l\delta_{jk}\delta_{im} - \\
&\qquad\qquad - 2\gamma^l\delta_{ik}\delta_{jm} + 2\gamma^j\delta_{il}\delta_{km} - 2\gamma^i\delta_{jl}\delta_{km}\Big\}_{\alpha\beta} \\
&\quad + \frac{i}{48R}\mathrm{STr}\,\Big(\theta\gamma^{[kij]}\theta\;\theta\gamma^k\theta - \theta\gamma^{[ij]}\theta\;\theta\theta\Big) \\
M^{+-ijkl} &= \frac{1}{12R}\mathrm{STr}\,\Big(F_{ij}F_{kl} + F_{ik}F_{lj} + F_{il}F_{jk} + \theta\gamma^{[jkl}[X^{i]},\theta]\Big) \qquad (3.16)
\end{aligned}
$$

The hatted and tilded currents $\hat{T}^{IJ},\ldots$ are given by using the appropriate variables $\hat{X},\hat{\theta}$ or $\tilde{X},\tilde{\theta}$ in these expressions. Time derivatives are taken with respect to Minkowski time t. In these expressions we have used the definitions $F_{0i} = \dot{X}^i, F_{ij} = i[X^i, X^j]$. The order θ^4 term in T^{--} is given by (2.20) where K and L are replaced by X and θ.

A few comments may be helpful regarding the derivation of the currents (3.16). The purely bosonic contributions to these tensors are those given in [30]. To determine the fermionic contributions from (2.21) it is helpful to write $C = C_{bos} + C_f$ (for $C = T$, J or M, suppressing tensor indices in this schematic discussion), where C_{bos} is the set of purely bosonic terms in C and C_f contains all other terms. If the matrix theory potential contains the interactions (3.13), (3.14), and (3.15), then it must contain terms corresponding to $\hat{C}_{bos}\tilde{C}_f$, $\hat{C}_f\tilde{C}_{bos}$, and $\hat{C}_f\tilde{C}_f$ in addition to the purely bosonic terms $\hat{C}_{bos}\tilde{C}_{bos}$.

The terms $\hat{C}_{bos}\tilde{C}_f$ and $\hat{C}_f\tilde{C}_{bos}$, in particular, each have one trace with purely bosonic fields and another trace which contains fermion fields. If we isolate such terms in the matrix theory effective potential, we can simply read off all the components of T_f, J_f, and M_f. For example, to determine T_f^{-i} we collect all terms in the leading order potential (2.21) of the form $\mathrm{Tr}\,(-i\dot{\hat{X}}_i)\mathrm{Tr}\,(\tilde{A}_i)$ and identify $\tilde{T}_f^{-i}$ as the sum of all the $\mathrm{Tr}\,(\tilde{A}_i)$ terms appearing (with an overall coefficient determined by (3.13)). Note that most of the bosonic currents have more than one term, so that this presciption is only consistent if each bosonic term in a given current couples to exactly the same set of terms containing fermions. The Matrix theory potential we have calculated passes this non-trivial consistency check with the currents defined as above.

Note that, on first glance, it appears that the last two terms in (2.20) should yield terms not accounted for in the preceding analysis, where $\mathrm{Tr}\,(X_iX_j)$ couples to a four-fermion expression and $\mathrm{Tr}\,(X_i)$ couples to a four-fermion expression. Both of these four-fermion expressions vanish, however, by Fierz identities.

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Note also that only the component M^{+-ijkl} of the 5-brane current has been identified. Only this component appears in the supergravity interaction because 5-branes which are not wrapped around the longitudinal (x^-) direction decouple from the theory so that $M^{ijklmn} = M^{+ijklm} = 0$. This decoupling follows from the fact that 5-branes act as Dirichlet boundaries for membranes, and so do not appear as interacting transverse objects in the light-front formalism [16]. The remaining component M^{-ijklm} can be determined from current conservation [42], as we will discuss shortly.

To complete the demonstration that V_{gravity}, V_{electric}, and V_{magnetic} are completely reproduced in Matrix theory at leading order, we should show that the terms $\hat{T}_f\tilde{T}_f$, and the corresponding terms for J and M also appear in the matrix theory potential. These must appear in the decomposition of the matrix theory potential as terms with two fermions in each trace (since T, J, and M may have no free fermionic indices) and therefore come from the four-fermion terms (2.20).

We note here that any decomposition of a four-fermion term to two traces with two L matrices in each trace may be written using a Fierz identity in a form where both the traces have no free fermionic indices. We must therefore consider all possible decompositions of the four-fermion terms into pairs of traces of fermion bilinears and show that these agree with the expressions for $\hat{T}_f\tilde{T}_f$. Examining the list of moments, we see that all such terms take the form

$$Tr(\hat{L}\hat{\gamma}[\hat{K},\hat{L}])Tr(\hat{L}\tilde{\gamma}[\tilde{K},\tilde{L}]) \tag{3.17}$$

in which $\hat{\gamma}$ and $\tilde{\gamma}$ are antisymmetric products of the same number of gamma matrices.

In the decomposition of (2.20) to pairs of traces with two fermions in each trace, we indeed find terms of this form, though there is also the possibility of terms in which both factors of K appear in a single trace, the other trace being either

$$\mathrm{Tr}\,(L\gamma^{ij}L)$$

or

$$\mathrm{Tr}\,(L\gamma^{ijk}L)\,.$$

These two expressions are identified below as part of the higher moments $J^{+ij(k)}$ and $T^{+i(j)}$, and should not affect the interactions at $1/r^7$, so we expect that all such terms will vanish, leaving only the desired terms (3.17). The demonstration of this, and the verification that terms of the form (3.17) appear with the correct coefficients for their interpretation as $\hat{T}_f\tilde{T}_f$, will not be completed here, as it would involve a large amount of algebra and is not important for our main applications.

There are various checks that may be performed on the expressions we have found. For example, those tensors with a $+$ index correspond to the light-front time component of a conserved current integrated over all space, and therefore correspond to conserved charges of the theory. In particular, T^{++}, T^{+i}, and T^{+-} are longitudinal momentum, transverse momentum and energy; J^{+ij} and J^{+-i} are the transverse and longitudinal membrane (string) charges, and M^{+-ijkl} is the longitudinal fivebrane charge. These charges all appear in the fully extended eleven-dimensional supersymmetry algebra, and matrix theory expressions for these charges were previously identified in [16] by taking commutators of the matrix theory supersymmetry generators. Our expressions agree completely with that work, providing a check for all components involving a $+$. The remaining components of the currents can be checked by ensuring that they correspond to the zeroth moments of components of conserved currents in 11 dimensions. To perform this check it is necessary to have expressions for the first moments of the currents, which appear in the interaction potential at order $1/r^8$, to which we now turn.

3.3 Gravitational and 3-form interactions at order $1/r^8$

We now consider the terms in the matrix theory potential at order $1/r^8$ which correspond to bosonic interactions in the linearized theory. These terms arise from interactions involving first moments of the supergravity currents, as in (3.10). For purely bosonic backgrounds, these terms were studied in [40, 30], where matrix theory expressions for the first moments of the bosonic currents were calculated and shown to give precise agreement between matrix theory and supergravity for terms at this order. As an example of the terms which appear at order $1/r^8$, from expressions (3.13) and (3.10) we see that the potential due to the exchange of a graviton with zero longitudinal momentum contains terms of the form

$$V^{1/r^8}_{\text{gravity}} = -\frac{105 R^2 r_i}{4 r^9}\left(\hat{T}_{IJ}\tilde{T}^{IJ(i)} + \hat{T}^{IJ(i)}\tilde{T}_{IJ} - \frac{1}{9}(\hat{T}^{I(i)}{}_I\tilde{T}^J{}_J + \hat{T}^I{}_I\tilde{T}^{J(i)}{}_J)\right).$$

Writing the $1/r^8$ matrix theory potential (2.22) as the product of a hatted trace and a tilded trace, and comparing it to this expression as we did for the $1/r^7$ potential, we may now read off the expressions for the first moments $\tilde{T}^{IJ(i)}$.

We can perform this analysis for all of the first moments of the bosonic supergravity currents. For each current, we find that the first moment in the i direction has a set of terms arising from simply inserting a matrix X_i into the symmetrized trace expression for the zeroeth moment. Contributions of this type to first and higher moments of the currents were studied in [40, 30], and come from the terms in the Matrix theory potential with an insertion of $r \cdot K$ (the first term in (2.22)). There are additional contributions to the first moments from terms where the moment index i appears on a gamma matrix. It is a fairly straightforward exercise to derive the additional contributions to each of the first moments which are quadratic in fermions from (2.22). It is interesting to note that the last two terms in (2.22) decompose completely into interactions with first moments of the type we are interested in, while the second and third terms have no such interpretation. We will return to these other terms in the next subsection. Here, we list some of the θ^2

contributions to the higher moments. We give explicit expressions only for the set of terms we will need to check the results of the previous subsection.

$$\begin{aligned}
T_{\rm add}^{+i(j)} &= \frac{1}{8R}{\rm STr}\,\left(\theta\gamma^{[ij]}\theta\right)\\
T_{\rm add}^{+-(i)} &= \frac{1}{16R}{\rm STr}\,\left(-i\theta F_{kl}\gamma^{[kli]}\theta+2i\theta\dot{X}_l\gamma^{[li]}\theta\right)\\
J_{\rm add}^{+ij(k)} &= \frac{i}{48R}{\rm STr}\,\left(\theta\gamma^{[ijk]}\theta\right)\\
J_{\rm add}^{+-i(j)} &= \frac{1}{48R}{\rm STr}\,\left(i\theta\dot{X}_k\gamma^{[kij]}\theta+i\theta F_{ik}\gamma^{[kj]}\theta\right)\\
M_{\rm add}^{+-ijkl(m)} &= -\frac{i}{16R}{\rm STr}\,\left(\theta F^{[jk}\gamma^{il]m}\theta\right).
\end{aligned}$$

These first moments have no contribution quartic in the fermion backgrounds. Note that the expression given here for $T^{i(j)}$ is responsible for a "spin" contribution to the angular momentum

$$\mathcal{J}^{ij}=T^{+i(j)}-T^{+j(i)}\,.$$

This generalizes the calculation in [20] where the analogous computation was done in the abelian case of 1×1 matrices.

We can now use these expressions for the first moments of the currents to check that the expressions (3.16) for the zeroth moments of the currents are correct. In [42] it was shown that conservation of the supergravity stress-energy tensor, membrane current, and fivebrane current implies relations among the matrix theory quantities identified with spatial moments of these currents. In particular, for a current C^I, agreement with the supergravity conservation relations implies that

$$\partial_t C^{+(i_1\cdots i_n)}=C^{i_1(i_2\cdots i_n)}+\cdots+C^{i_n(i_1\cdots i_{n-1})}\,.$$

In [42], these relations as they apply to T, J, and M were shown to hold for a bosonic background as exact matrix theory identities at finite N. Since we expect this to remain true for a general background, these relations should provide checks on the components of the currents we have determined with fermionic backgrounds.

In particular, we find for the case $n=1$ that the relations for the stress-energy tensor, membrane current, and fivebrane current are

$$\begin{aligned}
\partial_t T^{++(i)} &= T^{+i}\\
\partial_t T^{+i(j)} &= T^{ij}\\
\partial_t T^{+-(i)} &= T^{-i}\\
\partial_t J^{+ij(k)} &= J^{ijk}\\
\partial_t J^{+-i(j)} &= J^{-ij}\\
\partial_t M^{+-ijkl(m)} &= M^{-ijklm}\\
\partial_t M^{+ijklm(n)} &= M^{ijklmn}\,.
\end{aligned}\tag{3.18}$$

Thus, all components with a spatial index may be checked by comparing with the time derivative of the first spatial moment of the conserved charges previously checked. Note that none of the moments on the left hand sides of these relations contain four-fermion terms, so that we have exact expressions for both sides of each of these equations. The first five of the relations in (3.18) can be verified with some straightforward but tedious algebra. Note that the term M^{-ijklm}, which appears in the penultimate expression, could not be identified from the interaction potential, since it couples to the transverse fivebrane charge M^{+ijklm}, whose matrix theory expression appears to vanish. The conservation relation therefore serves to determine this component, as was done for its bosonic part in [42]. We find that

$$M^{-ijklm} = \frac{5}{4R}\mathrm{STr}\left(\dot{X}_{[i}F_{jk}F_{lm]} - \frac{1}{3}\theta\dot{X}^{[i}\gamma^{jkl}[X^{m]},\theta] - \frac{1}{6}\theta F^{[ij}\gamma^{klm]}\gamma^{i}[X^{i},\theta]\right).$$

The final expression in (3.18) is consistent with the statement that in the light-front theory $M^{ijklmn} = M^{+ijklm} = 0$ which, as mentioned above, arises from the fact that transverse M5-branes decouple in the light-front theory.

We have now performed an explicit check on all components of the stress energy tensor, membrane current, and fivebrane current given in (3.16) except for T^{--}. However, T^{--} is exactly the expression for the complete Matrix theory action at $1/r^7$, with K and L replaced by X and θ (since T^{++} is a constant). Since this is the expression from which we derived all of the other components, it must be correct up to terms which would not contribute to any of the other components. Such terms are four-fermion terms with all factors of K_i appearing in commutators with Ls. Thus, these are the only terms in the currents we have calculated for which we do not have an independent check. In principle these terms could be checked by verifying that they decompose correctly into the product $\hat{C}_f\tilde{C}_f$ of two-fermion terms we have explicitly computed. We have not performed this check in detail.

3.4 Membrane-fivebrane and "zerobrane-sixbrane" interactions

We have now given a complete interpretation in terms of supergravity interactions of all terms which appear in a decomposition of the leading-order interaction potential (2.21) into currents with purely bosonic indices. We have also interpreted a subset of the analogous terms in the potential (2.22) at order $1/r^8$. We have shown that the $r \cdot K$ terms and the last two terms in (2.22) have supergravity interpretations and correspond to first multipole corrections to (3.4)–(3.6), coupling the first spatial moment of each current to the zeroeth moment. The remaining terms in (2.22) have not yet been given an interpretation in terms of supergravity. We can rewrite these terms using equation (B.2) from the appendix as

$$\frac{35\, r_i}{256\, r^9}\epsilon^{ijklmnpqr}\mathrm{STr}\left(F_{jk}F_{lm}F_{np}\dot{K}_qK_r - L\,\dot{K}_jF_{kl}F_{mn}\gamma^{[pqr]}L + \frac{i}{2}L\,F_{jk}F_{lm}F_{np}\gamma^{[qr]}L\right). \tag{3.19}$$

We now consider all possible ways of decomposing this into a hatted trace times a tilded trace. Guided by the definitions of the moments above, we find that the decomposition gives a set of terms which can be identified as

$$\begin{aligned} V_{2-5} &= \frac{63R^2}{32}\frac{r^i}{r^9}\epsilon^{ijklmnpqr}\left(\hat{J}^{+jk}\tilde{M}^{-lmnpq(r)} + \hat{J}^{+jk(l)}\tilde{M}^{-mnpqr} + (\hat{}\leftrightarrow\tilde{})\right) + \\ &\quad + \frac{105R^2}{32}\frac{r^i}{r^9}\epsilon^{ijklmnpqr}\left(\hat{J}^{jkl}\tilde{M}^{+-mnpq(r)} + \hat{J}^{jkl(m)}\tilde{M}^{+-npqr} + (\hat{}\leftrightarrow\tilde{})\right) \\ &= \frac{7R^2}{64}\frac{r^I}{r^9}\epsilon_{IJKLMNPQRST}\left(\hat{J}^{JKL}\tilde{M}^{MNPQSR(T)} + \hat{J}^{JKL(M)}\tilde{M}^{NPQSRT} + (\hat{}\leftrightarrow\tilde{})\right). \end{aligned} \tag{3.20}$$

There are additional terms in the decomposition of (3.19) which do not have a simple interpretation in terms of the moments of the currents we have already discussed. These terms can be written in the form

$$\begin{aligned} V_{0-6} &= \frac{5R^2}{256}\frac{r^i}{r^9}\epsilon^{ijklmnpqr}\left(\hat{T}^{++}\tilde{S}^{jklmnpq(r)} + \hat{T}^{++(j)}\tilde{S}^{klmnpqr} + (\hat{}\leftrightarrow\tilde{})\right) + \\ &\quad + \frac{35R^2}{256}\frac{r^i}{r^9}\epsilon^{ijklmnpqr}\left(\hat{T}^{+j}\tilde{S}^{+klmnpq(r)} + \hat{T}^{+j(k)}\tilde{S}^{+lmnpqr} + (\hat{}\leftrightarrow\tilde{})\right) \\ &= \frac{5R^2}{256}\frac{r^a}{r^9}\epsilon_{abcdefghij}\left(\hat{T}^{+b}\tilde{S}^{cdefghi(j)} + \hat{T}^{+b(c)}\tilde{S}^{defghij} + (\hat{}\leftrightarrow\tilde{})\right), \end{aligned} \tag{3.21}$$

where we have defined the currents

$$\begin{aligned} S^{+ijklmn} &= \frac{1}{R}\mathrm{STr}\left(F_{[ij}F_{kl}F_{mn]}\right) \\ S^{+ijklmn(p)} &= \frac{1}{R}\mathrm{STr}\left(F_{[ij}F_{kl}F_{mn]}X_p - \theta F_{[ij}F_{kl}\gamma_{mnp]}\theta\right) \\ S^{ijklmnp} &= \frac{7}{R}\mathrm{STr}\left(F_{[ij}F_{kl}F_{mn}\dot{X}_{p]}\right) \\ S^{ijklmnp(q)} &= \frac{7}{R}\mathrm{STr}\left(F_{[ij}F_{kl}F_{mn}\dot{X}_{p]}X_q - \theta\,\dot{X}_{[i}F_{jk}F_{lm}\gamma_{npq]}\theta + \frac{i}{2}\theta\,F_{[ij}F_{kl}F_{mn}\gamma_{pq]}\theta\right). \end{aligned} \tag{3.22}$$

We will now discuss the supergravity interpretation of the terms (3.20) and (3.21). We begin with the potential V_{2-5} in (3.20). It is clear that this corresponds to a membrane-5-brane interaction potential. Indeed, we expect to see such interactions since both the membrane and the 5-brane couple to the 11D supergravity 3-form field A_{IJK}. It is fairly straightforward to see that, up to an overall multiplicative constant, (3.20) is precisely the right form for this interaction, as we now discuss in more detail.

Matrix theory interactions between a membrane and 5-brane were first discussed by Berkooz and Douglas in [31], where a 5-brane was put in as a static background for a dynamical matrix membrane. They showed that the fermion 0-modes corresponding to strings stretching between the membrane and 5-brane experience a correct Berry's phase shift when the membrane is carried around the 5-brane.

A configuration containing a membrane and a 5-brane which are both of infinite extent is analogous to a configuration in 4D electromagnetism containing both an electrically charged particle and a magnetic monopole. In principle, one might hope to describe an effective action for the electrically charged particle in the long-range field produced by the monopole. The vector potential A^μ around a monopole is not single-valued, however, and has a Dirac singularity. As a result, it is not possible to describe an interaction between electrically and magnetically charged particles in classical electromagnetism in terms of an effective potential. Similarly, it is not possible to construct an effective potential describing M-theory or matrix theory interactions between infinite membranes and 5-branes, so a study of fermion 0-modes is necessary to understand the interactions between such objects.

In this work, however, we are concerned with long-range interactions between finite-sized objects in matrix theory. Such objects cannot have net membrane or 5-brane charge. This can be seen directly from (3.16), where the membrane charge J^{+ij} and the 5-brane charge M^{+-ijkl} vanish for finite N. Finite N matrix theory configurations can, however, have nonzero moments of the membrane and 5-brane charges. Simple examples are given by the membrane with S^2 geometry and the L5-brane with S^4 geometry studied in [38, 43]. Unlike the interaction between an infinite membrane and an infinite 5-brane, an interaction between either brane and any configuration with higher moments of the other brane charge *can* be described by an effective potential. This is analogous to an interaction between an electron and a particle with a magnetic dipole moment. The gauge field A_μ around the dipole is single valued, so that the interaction can be written in the form $\int A^\mu j_\mu$ where j_μ is the 4-current of the electron.

The "electric" interactions between a pair of matrix theory configurations with nonzero moments of membrane charge are described by higher-order generalizations of (3.14). Although the first nonvanishing term of this form in the interaction potential between a pair of finite N configurations appears at order $1/r^9$, the expression (3.14) which formally appears in the potential at order $1/r^7$ is useful because it implies the precise forms of all the higher-moment electric interactions, as discussed in [30]. This is precisely analogous to the story in the case of membrane-5-brane interactions. The leading interaction (3.20) formally appearing in the long-range potential at order $1/r^7$ vanishes for a pair of finite-size configurations. Nonetheless, it correctly encodes all the higher-moment membrane-5-brane interactions which do not vanish.

The fact that (3.20) has precisely the right form for a membrane-5-brane interaction potential can be seen by a simple analogy with the electron-magnetic dipole case mentioned above. It may be helpful to review this situation explicitly. The long-range field of a magnetic monopole can be encoded in a dual vector potential

$$A_0^D = -g/r\,,$$

where g is the magnetic charge. This is related to a dual field strength $F^D_{\mu\nu} = \partial_\mu A^D_\nu - \partial_\nu A^D_\mu$ with nonzero components

$$F^D_{0i} = -\frac{gr^i}{r^3},$$

which is related through $F = {}^*F^D$ to the usual field strength of electromagnetism. This field has nonzero components

$$F_{ij} = -\frac{g\epsilon_{ijk}r^k}{r^3}$$

and cannot be written as the field strength of a single-valued potential A_μ. On the other hand, the field strength around a magnetic dipole is found by taking the derivative of the above expressions in the direction of the dipole d^i where $m^i = gd^i$. This gives

$$F_{ij} = -\frac{\epsilon_{ijk}m^k}{r^3} + \frac{3\epsilon_{ijl}r^l(r \cdot m)}{r^5} = \partial_i A_j - \partial_j A_i\,, \tag{3.23}$$

where the vector potential A_μ has nonzero components

$$A_i = \frac{\epsilon_{ijk}d^j r^k}{r^3}\,. \tag{3.24}$$

The generalization to the membrane-5-brane situation is straightforward. The long-range field around a longitudinal 5-brane is described by a dual potential (dropping overall constants)

$$A^D_{+-ijkl} = \int d^4\xi \frac{M^{+-ijkl}(\xi)}{r^7}$$

given by an integral over the local L5-brane charge. A finite volume configuration has vanishing net 5-brane charge but may have a nontrivial first moment $M^{+-ijkl(m)}$ which gives rise to a long-range field strength. $F = {}^*F^D$ which is the curvature of a well-defined 3-form field

$$A_{ijk} = \epsilon_{ijklmnpqs}\frac{M^{+-lmnp(q)}r^s}{r^9}\,.$$

If we consider a probe membrane in this long-range field, we find an effective potential described by the first term in the second line of (3.20). The other terms in (3.20) correspond in a similar fashion to the action for a probe 5-brane moving in the long-range field produced by a configuration with a nonzero first moment of the membrane current, and to terms related to these two by galilean invariance.

Thus, we see that (3.20) precisely encodes the form of interaction we expect to see at leading order between a pair of finite size objects in light-front supergravity. To connect this situation with the general discussion in section 3.1 in which classical supergravity interactions were described in terms of two-point functions of the propagating fields, we can consider a term coupling the membrane current and 5-brane

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current which appears in the effective action analogous to the expressions (3.4)–(3.6) for the interactions previously considered. The sources in (3.1) give rise to terms in the effective action

$$V_{\rm EM} = \int d^{11}x \int d^{11}y \hat{J}^{IJK}(x) \langle A_{IJK}(x) A^D_{PQRSTU}(y)\rangle \tilde{M}^{PQRSTU}(y) + (\hat{\ } \leftrightarrow \tilde{\ }) \,. \quad (3.25)$$

It is not clear that the expectation values written here make sense in a quantum theory, since they require a definition of the six form field as an operator written in terms of creation and annihilation operators for the basic supergravity fields. We are interested in terms with at least one higher moment, however, which give rise to propagators such as

$$\langle A_{IJK}(x) \partial_N A^D_{PQRSTU}(y)\rangle \sim \langle A_{IJK}(x) \epsilon^{GHLMNPQRSTU} \partial_G A_{HLM}(y)\rangle \,, \quad (3.26)$$

which can be understood in terms of the original 3-form propagator. This analysis again leads to a term of the form (3.20) in the supergravity potential.

We now return to the terms (3.21) which describe long-range interactions between gravitons and a new set of currents defined in (3.22). In the language of type IIA string theory, where we have a system of N 0-branes, the current $S^{+ijklmn}$ gives the 6-brane charge of the 0-brane configuration described by the matrices X^i. This is simply the T-dual of the statement that $\int F \wedge F \wedge F$ on a 6-brane counts 0-brane charge [13]. Given this interpretation, it is clear from current conservation and 10-dimensional symmetry that the remaining components of S correspond to other integrated components of the IIA 6-brane current. In (3.21) we see a coupling between the 6-brane current and a vector T^{+a} in the 10D IIA theory corresponding to the integrated 0-brane current. Of course, in type IIA string theory we expect to see precisely such a coupling. The 0-brane and 6-brane couple electrically and magnetically to the R-R 1-form field in IIA. The interaction between 0-branes and an infinite 6-brane is analogous to the interaction between an infinite membrane and 5-brane. Such an interaction cannot be described by an effective potential, but can be seen from the fermion 0-modes in the system, as in the case of the membrane-5-brane system. This leading-order interaction between 0-branes and an infinite 6-brane was studied in [44, 45, 46, 47, 48, 32].

The 6-brane charge of any set of matrices with finite size N must vanish, just as the membrane and L5-brane charge vanish. We therefore expect that the interactions between moments of the 6-brane current of one finite-size configuration and the 0-brane current of a second configuration should be described by a potential which has precisely the form (3.21), based on considerations precisely analogous to those above in the membrane-5-brane case. This 0-6 interaction, however, is completely described in terms of 10-dimensional currents. Unlike all the other interactions we have found it cannot be expressed in terms of a simple interaction potential with

11-dimensional invariance. This is a consequence of the fact that the IIA 6-brane has a more subtle connection with the 11-dimensional M-theory than the 0-, 2- and 4-branes, which correspond to extended objects in M-theory of dimension 0, 2 and 5. The infinite flat 6-brane as a IIA background can be interpreted in 11D supergravity as a Kaluza-Klein monopole background metric. The interpretation of a finite size IIA object with local 6-brane charge is somewhat less clear. Presumably it corresponds to a more complicated background metric in 11 dimensions. Thus, we have a physical interpretation of the interactions in (3.21) in 10 dimensions which is somewhat more transparent than the 11-dimensional interpretation, in which the matrix theory configuration with 6-brane charge seems to correspond to a nontrivial background geometry in M-theory.

3.5 Supercurrent interactions

In the previous subsections, we have considered all terms in the decomposition of the matrix theory action at orders $1/r^7$ and $1/r^8$ into two traces with no free fermionic indices and shown that these terms correspond precisely with the leading and subleading order supergravity interactions due to the exchange of a graviton or three form quantum. The only remaining terms in the decomposition of the matrix theory potential contain a single free fermionic index in each trace. (We argued above that terms with two fermionic indices in each trace could be rewritten using a Fierz identity into a form with no free fermionic indices in either trace.) We would like to identify these terms with interactions of the form (3.7) arising from the exchange of a gravitino. It may seem unusual to consider an interaction between two fixed sources in which a fermionic gravitino is exchanged, as this would seem to imply a change of the statistics of the two objects in the course of the interaction. This can be understood for classical objects, however in terms of a probe-source picture where one object probes the long-range gravitino field ψ_J produced by a distant source, giving an interaction potential of the form $\int \bar{\psi}_J S^J$ where S^J is the fermionic part of the supercurrent of the probe. Interactions of this form have also been discussed recently in [49].

To see the form of the leading-order supergravity interaction due to gravitino exchange, we must explicitly determine the gravitino propagator. In the supergravity action, the quadratic terms are

$$\Gamma_{\Psi\Psi} = -\frac{1}{2}\bar{\Psi}_I \Gamma^{IJK} \partial_K \Psi_J \,.$$

Supergravity is invariant under local supersymmetry transformations which affect the gravitino field, so as for the case of a local gauge symmetry, we must add gauge fixing terms to the action in order to quantize the theory [50]. These affect the form of the propagator, but as for the case of graviton and three-form exchange, there appears to be a natural choice which makes the correspondance with Matrix theory most clear. In this case, we choose a gauge-fixing term

$$\Gamma_{\text{fix}} = \frac{9}{8}\bar{\Psi}_I \Gamma^I \partial\!\!\!/ \Gamma^J \Psi_J \,.$$

This choice is analogous to Feynman gauge in electromagnetism, ensuring that the momentum space propagator contains only terms with a single k in the numerator.

The propagator $\Delta^{\alpha\beta}_{IJ}$ is then determined by

$$\left(-\frac{1}{2}\Gamma^{IJL}\partial_L + \frac{9}{8}\Gamma^I \partial\!\!\!/ \Gamma^J\right)\Delta_{JK}(x-y) = \delta^I_K \delta(x-y)\,.$$

As with the graviton and three form propagators, we restrict to the x^- independent term corresponding to exchange of gravitinos with zero longitudinal momentum, and we find

$$\Delta_{IJ} = \langle \Psi_I(x)\bar{\Psi}_J(y)\rangle \propto (\Gamma_J \partial\!\!\!/ \Gamma_I - 7\delta_{IJ}\partial\!\!\!/)\delta(x^+ - y^+)\frac{1}{|\vec{x}-\vec{y}|^7}\,.$$

We now insert this expression into (3.7) to determine the leading-order terms in the supergravity potential due to gravitino exchange. We find an interaction proportional to

$$\int dx dy \bar{\tilde{S}}^I(x)\left\{\Gamma_J\Gamma^K\Gamma_I - 7\delta_{IJ}\Gamma^K\right\}\partial_K \delta(x^+ - y^+)\frac{1}{|\vec{x}-\vec{y}|^7}\tilde{S}^J(y)\,.$$

The only terms at order $1/r^7$ come from choosing ∂_K to be a time derivative ($K = +$), since the x^- derivative vanishes, while the transverse spatial derivatives acting on $1/r^7$ result in an additional power of $1/r$. To find the explicit form of these leading-order interactions, we expand the currents as distributions (3.9) and find

$$V^{\text{super}}_{1/r^7} \propto \frac{R^2}{r^7}\int d\tau \bar{\tilde{S}}^I\{\Gamma_J\Gamma_-\Gamma_I - 7\delta_{IJ}\Gamma_-\}\dot{\tilde{S}}^J\,,$$

where there is an overall rational coefficient we have not calculated. To connect with our Matrix theory potential, we write S in terms of sixteen-component spinors q^I and Q^I as

$$S^I = \begin{pmatrix} q^I \\ Q^I \end{pmatrix}, \qquad \bar{S}^I = (S^I)^\dagger \Gamma^0$$

and choose gamma matrices

$$\Gamma^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix}, \qquad \Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma^{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(Note that the convention used in this section for 11D gamma matrices is different from the convention (2.13) used elsewhere for 10D gamma matrices.) We have then

$$\Gamma^+ = -\Gamma_- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \qquad \Gamma^- = -\Gamma_+ = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix}$$

and we find

$$V^{\text{super}}_{1/r^7} \propto \frac{R^2}{r^7}\Big(\hat{q}^+\,\partial_t\,\left(7\tilde{q}^- - \sqrt{2}\gamma^i\tilde{Q}^i\right) + \left(7\hat{q}^- - \sqrt{2}\hat{Q}^i\gamma^i\right)\partial_t\,\tilde{q}^+ - 2\hat{Q}^+\,\partial_t\,\tilde{Q}^+ + \\ + \hat{q}^i\gamma^i\,\gamma^j\,\partial_t\,\tilde{q}^j - 9\hat{q}^i\,\partial_t\,\tilde{q}^i\Big)\,. \tag{3.27}$$

Before comparing this with the matrix theory potential, we recall a few physical facts about the supercurrent $S^I(x)$. Firstly, since the gravitino is massless and has a vector index, the source S^I must obey the constraint

$$\partial_I S^I(x) = 0\,.$$

Thus, the zeroeth moment S^+, and therefore the charges q^+ and Q^+ are conserved in light-front time. These quantities are the supersymmetry generators of the theory, whose matrix theory counterparts are known to be [16]:

$$q^+ = \frac{c_1}{R}\mathrm{Tr}\,(\theta)\,, \qquad Q^+ = \frac{c_2}{R}\mathrm{Tr}\,\left(\left(\dot{X}^i\gamma_i + \frac{1}{2}F_{ij}\gamma^{ij}\right)\theta\right). \tag{3.28}$$

One may easily check that $\dot{q}^+ = \dot{Q}^+ = 0$. The overall numerical coefficient multiplying the currents in this section depends on the conventions chosen to define the current S, so we will not keep careful track of it. However, we will be able to fix the normalization on the other components of the current using the constants c_1 and c_2 appearing in these expressions.

Looking back at the potential (3.27) we now see that last two terms on the first line vanish, while the first is a total derivative and therefore does not correspond to a physical interaction. The remaining terms involve only the component q^i.

Turning now to the matrix theory potential, we expect that with the proper identification of matrix theory quantities for the supercurrents, the expression (3.27) should reproduce all terms in the decomposition of the $1/r^7$ potential with two fermionic traces. It is not difficult to check that using the matrix theory equations of motion, such terms may be rewritten as (ignoring four fermion terms)

$$V_{\text{mat}} \propto \mathrm{STr}\,\left(\hat{\theta}\right)\partial_\tau\mathrm{STr}\,\left(\left\{-\frac{1}{4}\gamma^{[ijkl]}F_{ij}F_{kl} + \frac{1}{2}F_{ij}F_{ji} - 2i\dot{\tilde{X}}_iF_{ij}\gamma^j + \dot{\tilde{X}}_i\dot{\tilde{X}}_i\right\}\tilde{\theta}\right) + \\ + \frac{2i}{9}\mathrm{STr}\,\left(\hat{\theta}\left\{i\dot{\hat{X}}_i + F_{ij}\gamma^j\right\}\right)\left(\gamma^i\gamma^k - 9\delta_{ik}\right)\partial_\tau\mathrm{STr}\,\left(\left\{i\dot{\tilde{X}}_k + F_{kl}\gamma^l\right\}\tilde{\theta}\right) + \{\hat{\ }\leftrightarrow\tilde{\ }\}$$

This matches exactly with the nonvanishing terms in the supergravity potential (3.27), provided we identify (in Minkowski notation)

$$q^i = \frac{c_1}{R}\mathrm{STr}\,\left(\{\dot{X}_i - F_{ij}\gamma^j\}\theta\right)$$

and

$$\left(7q^- - \sqrt{2}\gamma^iQ^i\right) = \frac{9c_1}{2R}\mathrm{STr}\,\left(\left\{-\frac{1}{4}\gamma^{[ijkl]}F_{ij}F_{kl} + \frac{1}{2}F_{ij}F_{ji} + 2\dot{X}_iF_{ij}\gamma^j - \dot{X}_i\dot{X}_i\right\}\theta\right). \tag{3.29}$$

(with overall coefficients which are determined relative to those of (3.28) below.) Note that the latter definition comes from identifying a total derivative term in the supergravity potential with a total derivative in the Matrix theory potential. For more solid evidence of the this identification, it is necessary to look at higher order interactions at $1/r^8$. We also note that Q^+ and Q^- do not appear in the potential at all, while q^- and Q^i appear only in the combination $(7q^- - \sqrt{2}\gamma^i Q^i)$. To make explicit identifications of these terms by comparing interaction potentials, one would again have to look at terms of order $1/r^8$.

As a check on the q^i components we have identified and to identify the components Q^i and q^-, we can use the conservation relation $\partial_I S^I$ mentioned above. As for the bosonic currents, this implies that the spatial moments of q and Q are related to the time derivative of moments with a + index. In particular, we expect

$$q^i = \partial_t q^{+(i)}, \qquad Q^i = \partial_t Q^{+(i)}.$$

Though we have not identified the first moments $q^{+(i)}$ or $Q^{+(i)}$ through an analysis of the potential terms as with the bosonic currents, we expect these to have terms given by the insertion of X^i into the zeroth moments plus possibly other terms with the i index appearing on a gamma matrix. Assuming no such additional terms are present, we would have

$$\begin{aligned} q^i = \partial_t q^{+(i)} &= \partial_t \frac{c_1}{R} \mathrm{STr}\,(X_i \theta) \\ &= \frac{c_1}{R} \mathrm{STr}\left(\left\{\dot{X}^i - F_{ij}\gamma^j\right\}\theta\right). \end{aligned}$$

This agrees with our previous expression for q^i and fixes the normalization relative to q^+. For the Q components, again assuming that all terms in $Q^{+(i)}$ are given by X^i insertions into Q^+ we have

$$\begin{aligned} Q_i = \partial_t Q^{+(i)} &= \partial_t \frac{c_2}{R} \mathrm{STr}\left(X^i\left\{\dot{X}^j\gamma_j + \frac{1}{2}F_{jk}\gamma^{jk}\right\}\theta\right) \\ &= \frac{c_2}{2R}\,\mathrm{STr}\left(\left\{2\dot{X}_i\dot{X}^j\gamma_j + \dot{X}_i F_{jk}\gamma^j\gamma^k + 2\dot{X}^j F_{li}\gamma^j\gamma^l + F_{jk}F_{li}\gamma^j\gamma^l\gamma^k\right\}\theta\right). \end{aligned}$$

It is possible that there are additional terms in Q^i arising from extra terms in the moment $Q^{+(i)}$ that we have not accounted for, however if this is not the case, we may use our identification in expression (3.29) to define q^-. A careful analysis of the potential at $1/r^8$ would provide a check on our definitions of q^- and Q^i as well as an identification of the component Q^-. Analysis of these higher order terms should also fix the coefficient c_2 relative to c_1, not possible using only the $1/r^7$ potential.

We have now given an interpretation of all terms at leading and subleading orders in the general matrix theory potential between two isolated systems and found within this potential all interactions expected from the exchange of a single quantum in supergravity.

3.6 Summary of supergravity currents

We summarize here our results for the matrix form of the supergravity currents. The integrated stress-energy tensor, membrane current and 5-brane current are given in matrix theory by

$$
\begin{aligned}
T^{++} &= \frac{1}{R}\mathrm{STr}\ (\mathbb{1}) \\
T^{+i} &= \frac{1}{R}\mathrm{STr}\ \left(\dot{X}_i\right) \\
T^{+-} &= \frac{1}{R}\mathrm{STr}\ \left(\frac{1}{2}\dot{X}_i\dot{X}_i + \frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\theta\gamma^i[X^i,\theta]\right) \\
T^{ij} &= \frac{1}{R}\mathrm{STr}\ \left(\dot{X}_i\dot{X}_j + F_{ik}F_{kj} - \frac{1}{4}\theta\gamma^i[X_j,\theta] - \frac{1}{4}\theta\gamma^j[X_i,\theta]\right) \\
T^{-i} &= \frac{1}{R}\mathrm{STr}\ \left(\frac{1}{2}\dot{X}_i\dot{X}_j\dot{X}_j + \frac{1}{4}\dot{X}_iF_{jk}F_{jk} + F_{ij}F_{jk}\dot{X}_k\right) - \\
&\quad -\frac{1}{4R}\mathrm{STr}\ \left(\theta_\alpha\dot{X}_k[X_m,\theta_\beta]\right)\{\gamma^k\delta_{im} + \gamma^i\delta_{mk} - 2\gamma^m\delta_{ki}\}_{\alpha\beta} - \\
&\quad -\frac{1}{8R}\mathrm{STr}\ (\theta_\alpha F_{kl}[X_m,\theta_\beta])\{\gamma^{[iklm]} + 2\gamma^{[lm]}\delta_{ki} + 4\delta_{ki}\delta_{lm}\}_{\alpha\beta} + \\
&\quad +\frac{i}{8R}\mathrm{Tr}\,(\theta\gamma^{[ki]}\theta\ \theta\gamma^k\theta) \\
T^{--} &= \frac{1}{4R}\mathrm{STr}\ \left(F_{ab}F_{bc}F_{cd}F_{da} - \frac{1}{4}F_{ab}F_{ab}F_{cd}F_{cd} + \theta\Gamma^a\Gamma^b\Gamma^cF_{ab}F_{cd}D_a\theta + \mathcal{O}(\theta^4)\right) \\
J^{+ij} &= \frac{1}{6R}\mathrm{STr}\ (F_{ij}) \\
J^{+-i} &= \frac{1}{6R}\mathrm{STr}\ \left(F_{ij}\dot{X}_j - \frac{1}{2}\theta[X_i,\theta] + \frac{1}{4}\theta\gamma^{[ki]}[X_k,\theta]\right) \\
J^{ijk} &= \frac{1}{6R}\mathrm{STr}\ \left(\dot{X}_iF_{jk} + \dot{X}_jF_{ki} + \dot{X}_kF_{ij} - \frac{1}{4}\theta\gamma^{[ijkl]}[X_l,\theta]\right) \\
J^{-ij} &= \frac{1}{6R}\mathrm{STr}\ \left(\dot{X}_i\dot{X}_kF_{kj} - \dot{X}_j\dot{X}_kF_{ki} - \frac{1}{2}\dot{X}_k\dot{X}_kF_{ij} + \frac{1}{4}F_{ij}F_{kl}F_{kl} + F_{ik}F_{kl}F_{lj}\right) + \\
&\quad +\frac{1}{24R}\mathrm{STr}\ \left(\theta_\alpha\dot{X}_k[X_m,\theta_\beta]\right)\Big\{\gamma^{[kijm]} + \gamma^{[jm]}\delta_{ki} - \\
&\qquad\qquad - \gamma^{[im]}\delta_{kj} + 2\delta_{jm}\delta_{ki} - 2\delta_{im}\delta_{kj}\Big\}_{\alpha\beta} + \\
&\quad +\frac{1}{8}\mathrm{STr}\,(\theta_\alpha F_{kl}[X_m,\theta_\beta])\Big\{\gamma^{[jkl]}\delta_{mi} - \gamma^{[ikl]}\delta_{mj} + 2\gamma^{[lij]}\delta_{km} + 2\gamma^l\delta_{jk}\delta_{im} - \\
&\qquad\qquad - 2\gamma^l\delta_{ik}\delta_{jm} + 2\gamma^j\delta_{il}\delta_{km} - 2\gamma^i\delta_{jl}\delta_{km}\Big\}_{\alpha\beta} + \\
&\quad +\frac{i}{48R}\mathrm{STr}\ \left(\theta\gamma^{[kij]}\theta\ \theta\gamma^k\theta - \theta\gamma^{[ij]}\theta\ \theta\theta\right) \\
M^{+-ijkl} &= \frac{1}{12R}\mathrm{STr}\ \left(F_{ij}F_{kl} + F_{ik}F_{lj} + F_{il}F_{jk} + \theta\gamma^{[jkl}[X^{i]},\theta]\right) \\
M^{-ijklm} &= \frac{5}{4R}\mathrm{STr}\ \left(\dot{X}_{[i}F_{jk}F_{lm]} - \frac{1}{3}\theta\dot{X}^{(i}\gamma^{jkl}[X^{m)},\theta] - \frac{1}{6}\theta F^{(ij}\gamma^{klm)}\gamma^i[X^i,\theta]\right) \\
M^{+ijklm} &= M^{ijklmn} = 0
\end{aligned}
$$

The first moments of these currents are given by

$$\begin{aligned} T^{IJ(i)} &= \mathrm{Sym}\,(T^{IJ}; X^i) + T^{IJ(i)}_{\rm add} \\ J^{IJK(i)} &= \mathrm{Sym}\,(J^{IJK}; X^i) + J^{IJK(i)}_{\rm add} \\ M^{IJKLMN(i)} &= \mathrm{Sym}\,(M^{IJKLMN}; X^i) + M^{IJKLMN(i)}_{\rm add}\,, \end{aligned} \tag{3.30}$$

where in particular

$$\begin{aligned} T^{+i(j)}_{\rm add} &= \frac{1}{8R}\mathrm{Tr}\,(\theta\gamma^{[ij]}\theta) \\ T^{+-(i)}_{\rm add} &= \frac{1}{16R}\mathrm{Tr}\,(-i\theta F_{kl}\gamma^{[kli]}\theta + 2i\theta\dot{X}_l\gamma^{[li]}\theta) \\ J^{+ij(k)}_{\rm add} &= \frac{i}{48R}\mathrm{Tr}\,(\theta\gamma^{[ijk]}\theta) \\ J^{+-i(j)}_{\rm add} &= \frac{1}{48R}\mathrm{STr}\,\left(i\theta\dot{X}_k\gamma^{[kij]}\theta + i\theta F_{ik}\gamma^{[kj]}\theta\right) \\ M^{+-ijkl(m)}_{\rm add} &= -\frac{i}{16R}\mathrm{STr}\,\left(\theta F^{[jk}\gamma^{il]m}\theta\right) \end{aligned}$$

are the two-fermion contributions to the first moments with a + index.

The fermionic components of the supercurrent (excluding possible three fermion terms) are given by

$$\begin{aligned} q^+ &= \frac{c_1}{R}\mathrm{Tr}\,(\theta) \\ q^i &= \frac{c_1}{R}\mathrm{STr}\,\left(\left\{\dot{X}^i - F_{ij}\gamma^j\right\}\theta\right) \\ Q^+ &= \frac{c_2}{R}\mathrm{STr}\,\left(\left(\dot{X}^j\gamma_j + \frac{1}{2}F_{ij}\gamma^{ij}\right)\theta\right) \\ Q^i &= \frac{c_2}{2R}\mathrm{STr}\,\left(\left\{2\dot{X}_i\dot{X}^j\gamma_j + \dot{X}_iF_{jk}\gamma^j\gamma^k + 2\dot{X}^jF_{li}\gamma^j\gamma^l + F_{jk}F_{li}\gamma^j\gamma^l\gamma^k\right\}\theta\right). \end{aligned}$$

As discussed above, the components q^- and Q^- as well as the ratio c_1/c_2 should follow from an analysis of the $1/r^8$ potential. The remaining overall coefficient depends on the conventions used for defining the supercurrent.

There is also a current for the IIA 6-brane, corresponding to a Kaluza-Klein monopole in 11 dimensions. This current has integrated components

$$\begin{aligned} S^{+ijklmn} &= \frac{1}{R}\mathrm{STr}\,\left(F_{[ij}F_{kl}F_{mn]}\right) \\ S^{+ijklmn(p)} &= \frac{1}{R}\mathrm{STr}\,\left(F_{[ij}F_{kl}F_{mn]}X_p - \theta F_{[ij}F_{kl}\gamma_{mnp]}\theta\right) \\ S^{ijklmnp} &= \frac{7}{R}\mathrm{STr}\,\left(F_{[ij}F_{kl}F_{mn}\dot{X}_{p]} + f.t.\right) \\ S^{ijklmnp(q)} &= \frac{7}{R}\mathrm{STr}\,\left(F_{[ij}F_{kl}F_{mn}\dot{X}_{p]}X_q - \theta\,\dot{X}_{[i}F_{jk}F_{lm}\gamma_{npq]}\theta + \frac{i}{2}\theta\,F_{[ij}F_{kl}F_{mn}\gamma_{pq]}\theta\right). \end{aligned} \tag{3.31}$$

4. Conclusions and further directions

In this paper we have calculated the most general form of the leading $1/r^7$ term in the two-body interaction potential between an arbitrary pair of objects in matrix theory. We included completely general bosonic and fermionic background matrices corresponding to arbitrary non-supersymmetric objects. We found that the general interaction potential decomposes into a sum of parts which can be identified with supergravity interactions between extended objects arising from the exchange of a single graviton, gravitino or 3-form quantum. This correspondence was made by giving precise matrix expressions for the stress-energy tensor, membrane current, 5-brane current and fermionic supercurrent components of an arbitrary matrix theory configuration, generalizing the results of [30] for purely bosonic backgrounds.

We calculated the general form of the subleading $1/r^8$ terms in the interaction potential to quadratic order in the fermion backgrounds. We found that beginning at this order a new set of terms appear in the potential corresponding to membrane-5-brane and 0-brane-6-brane interactions. These terms are analogous to the effective potential a probe electron would see in the long-range electromagnetic field of an object with nonzero magnetic multipole moments. We also derived explicit expressions for some of the fermionic contributions to the first moments of the gravity currents, generalizing the result for the spin contribution to angular momentum in [20].

Although we only explicitly discussed the $1/r^7$ and $1/r^8$ terms in the effective potential in this paper, the terms we calculated here correspond to an infinite series of higher-moment interaction terms which appear at all orders in $1/r$. As discussed in [40, 30], for every interaction term in the $1/r^7$ interaction potential, an infinite number of other terms can be produced by symmetrizing with an arbitrary number k of factors of $r \cdot K$ and including an appropriate combinatorial coefficient. In fact, most of the terms we discussed here formally vanish for finite size objects unless some higher moments are taken. For example, the "electric" membrane-membrane interaction vanishes out to order $1/r^9$ for finite N configurations, since a finite size object cannot have net membrane charge, but can have a dipole moment of membrane charge.

In addition to the higher-moment terms which arise in the one-loop potential from factors of $r \cdot K$ there will be terms of arbitrary order in the fermions which contribute to higher moments, generalizing the results in [20, 21, 22, 23, 24] and the general results to quadratic order in this paper. It would be nice to have some systematic understanding of the complete set of higher-order fermion terms.

The fact that there is an infinite family of terms in the matrix theory effective potential, including terms at all orders in $1/r$, which agree with supergravity even for finite N is rather remarkable. It suggests that it should be possible to extend to arbitrary backgrounds the nonrenormalization theorems of [18, 19, 51]. At present there seems to be very little understanding of why such nonrenormalization theorems may exist. This is clearly a crucial issue to understand better, however, as these nonrenormalization theorems lie at the heart of matrix theory's ability to reproduce 11-dimensional gravitational physics.

This work connects to a wide range of questions which are currently of interest in the study of matrix theory. We conclude this paper with a brief discussion of some ways in which this work connects to other ideas and some directions for further study. In particular, we discuss the connection of these results to supersymmetric nonabelian Born-Infeld theory, the extension of these results to higher dimensions, the extension of this analysis to the N-body problem in matrix theory, and a proposal for using the gravity currents defined here (and their generalizations) to formulate matrix theory in a general background geometry.

4.1 Supersymmetric nonabelian Born-Infeld theory

It has been suggested by a number of authors [33, 34, 35, 52] that the infinite series of terms in the 2-body matrix theory potential of the form F^{2k}/r^{7k-7} should naturally take the form of a nonabelian Born-Infeld action. In the case of two gravitons, the Born-Infeld theory is abelian and this is precisely the form of the effective potential expected from supergravity [17]. Similar arguments have been made in other situations where the theory is truly nonabelian. The symmetrized trace form of the F^4/r^7 term (2.14) agrees with the suggestion by Tseytlin [53] that the ordering problems inherent in defining a nonabelian Born-Infeld action should be resolved by symmetrizing the trace over all orderings of F's at each order (in fact, (2.14) was derived in [54] from string theory as part of the nonabelian Born-Infeld action). If it is true that the 2-body potential in matrix theory can be expressed in a nonabelian Born-Infeld form, it should follow that the complete set of terms at order $1/r^7$ should be a supersymmetric completion of the F^4/r^7 bosonic term. Although the exact form of the corrections to the tree-level supersymmetry transformations are not known, it was shown in [36] and [37] that the supersymmetric *abelian* Born-Infeld action is quite constrained at orders $F^2\bar{\theta}D\theta$ and $\bar{\theta}D\theta\bar{\theta}D\theta$. Comparing our results to these two papers provides a further check on our expressions for the two-fermion and four-fermion parts of the $1/r^7$ interaction. Comparing to [36], equation (7) we find perfect agreement. Setting $c_1 = c_3 = 0$ and $c_2 = -2$ in that equation through a field redefinition, and integrating by parts to put the derivative on the fermion field, we find that their expression agrees perfectly with the abelianization of the two-fermion terms in (2.21). (Note that our normalization of the fermion fields differs from theirs by a factor of 2). For the four-fermion terms, we recall that our action (2.20) contained $D = 10$ covariant terms as well as some non-covariant terms. We expect that the abelian version of our covariant terms should obey the constraints found in [37], and we find in fact that the $D = 10$ covariant terms in our action reduce exactly to the four-fermion term in equation (12) of [37]. Our action contains one additional non-covariant term in the abelian case, the first term in the fourth line of (2.20), but this part of the result would seem to be specific to $D = 1$.

4.2 Effective action in other dimensions

In performing our calculation of the effective action, we found that the form of the effective action in the $0 + 0$ dimensional theory was almost identical to our $0 + 1$ dimensional result. For bosonic and two fermion terms, we found that up to an overall constant at each order, the $0 + 0$ dimensional action is obtained simply by ignoring the terms with insertions of D_0^2 and replacing $\vec{K}^2$ with K^2 in the other non-covariant terms. This suggests that for other dimensions, the answer would also have an analogous form, generalizing the D_0^2 insertions we have found in the $(0+1)$-dimensional case to insertions of a covariant derivative squared in higher dimensions, and in the other non-covariant terms interpreting $\vec{K}^2$ to be $K_i K_i$ with $i \geq D-1$. It is not immediately clear how to generalize the non-covariant pieces in the four-fermion terms, however, since these do not appear to be simply related to any covariant terms. It may be that there is an alternate way of writing these terms that would make their generalization more transparent.

We note also that the results we have found are not really restricted in validity to the case of a block-diagonal background. If we were to compute the effective action in the usual sense with a general $(N \times N)$ background, we would use an action formally identical to (2.2), omitting the factors of r and interpreting Y and K as matrices in the adjoint of $\mathrm{U}(N)$. The calculation would then be the same as what we have done here, except that we would have only massless propagators and therefore we would have some divergent coefficients in place of powers of r. Thus, our result may alternately be interpreted as the full effective action with r acting as a regulator. In the $(0 + 1)$-dimensional theory, this allows us to interpret the one-loop potential as the linearized gravity self-interaction of a single matrix theory configuration. In higher-dimensional theories, the result can be interpreted as an effective action for excitations of the internal gauge fields as well as transverse scalars in the $\mathrm{U}(N)$ gauge theory on the world-volume of a system of parallel D-branes.

The generalization of our results for the fermionic terms in the effective potential might be of particular interest for the case $D = 4$, where the gauge theory in question is believed to correspond to IIB string theory on $AdS_5 \times S^5$ [55]. In this theory it is known that the supersymmetric one-loop effective potential at order F^4 is not renormalized [56, 57]. To our knowledge, the fermionic terms in this potential have not been written explicitly. The above discussion gives an explicit conjecture for the $F^2\bar{\theta}D\theta$ terms, although further work is needed to find the terms quartic in fermion backgrounds.

4.3 n-body interactions

As discussed in the introduction, there has been a great deal of work done on the problem of understanding how n-graviton interactions in supergravity can be reproduced by matrix theory calculations. In fact, there has been an extended controversy regarding the question of whether finite N matrix theory calculations can correctly reproduce the classical supergravity interactions between an arbitrary number of gravitons. It is now generally agreed that the first nontrivial term in the effective supergravity potential of a 3-graviton system can be reproduced by a 2-loop matrix theory calculation. This issue was discussed in [25, 26, 27] and was conclusively decided by the complete 2-loop calculation of Okawa and Yoneya in [28, 58]. Some terms describing n-graviton interactions were shown to be correctly reproduced by matrix theory in [29]. In the same paper, however, it was suggested that some terms appear in the 3-loop matrix theory description of a 4-body interaction which do not correspond to terms in the supergravity theory. It has been suggested that 4-body interactions may not be correctly reproduced from matrix theory based on other discrepancies found when matrix theory is considered on general non-flat backgrounds; we discuss the issue of general backgrounds further in the following subsection.

In general, the $(n-1)$st-order nonlinear effects of classical gravity can be seen most easily in an interacting system of n bodies. Clearly, achieving a better understanding of whether and how finite N matrix theory reproduces classical supergravity interactions between more than two objects is a crucial issue. Resolving this issue will help answer the questions of whether matrix theory is correct, and if it is correct whether it is possible to reproduce supergravity effects to an arbitrary degree of precision by choosing large but finite N, or whether there is some unexpected subtlety in the large N limit related to effects which appear as the size of a graviton bound state exceeds the scale of its separation from other objects in the theory.

While comparing amplitudes and potentials for n-graviton scattering is an interesting test of matrix theory, in general the resulting interactions are fairly constrained by the Galilean invariance of the light-front theory, and give essentially only some numerical checks for coefficients in a finite number of terms. (Although the structure found in [28] is indeed quite nontrivial.) Furthermore, single gravitons are BPS objects which preserve much of the supersymmetry of the theory, so a skeptic may argue that the agreement between matrix theory and supergravity for interactions between such systems is easily protected by supersymmetry and does not constitute a strong check of the theory. It would be a significantly more powerful check of the theory if it were possible to generalize the structural arguments used in this paper and in [30] to interactions between more than two objects. Even showing that the leading nonlinear term in a general 3-body interaction is correctly reproduced by supergravity using the definitions we have given of the stress-energy tensor and other supergravity currents would indicate that matrix theory completely encodes the 3-supergraviton vertex of the light-front theory. It is conceivable that this alone would be sufficient to give an inductive proof that matrix theory reproduces supergravity at all orders.

The results of this paper are an important intermediate step in the calculation of the minimal general 3-body interaction potential of matrix theory. Note that the backgrounds corresponding to n-body interactions are a special case of the background we have chosen. We did not assume anything about the two systems we have studied, so we could take either of our systems to contain two or more widely separated objects by choosing $\hat{X}$ or $\tilde{X}$ to be block diagonal. For example, if we take $\tilde{X}$ to itself be a block-diagonal matrix representing two matrix theory objects separated by a distance ρ which is large but still much less than r then we can calculate the terms in the effective potential at order $1/(r^7\rho^7)$ by first integrating out the off-diagonal fields corresponding to strings stretching between $\hat{X}$ and $\tilde{X}$ and then integrating out further fields which are off-diagonal in $\tilde{X}$. These terms should correspond to the leading nonlinear terms in the gravitational effective potential of the corresponding 3-body system. Such effects can be understood from one point of view as the effective action for a body $\hat{X}$ considered as a probe moving in the long-range gravitational field produced by $\tilde{X}$, where the terms of order $1/(r^7\rho^7)$ arise from the contribution to the long-range gravitational field which is quadratic in the sources encoded by $\tilde{X}$. This type of argument was used to show that certain terms in the 3-graviton potential are correctly reproduced by matrix theory in [27]. Furthermore, the problematic four graviton term discussed by Dine, Echols, and Gray in [29] is obtained in just this way by performing successive loop integrals, so it should also appear in our general framework.

There is one important subtlety here which must be considered in carrying out the calculation at higher-loop order. The K appearing in our result for the effective action is really a background field. Looking back at our definitions of the fields (2.1), we see that it is really off-diagonal fields in $\hat{Z}$ and $\tilde{Z}$, (rather than off diagonal fields in the background K) that we would integrate out to get an n-body interaction. In the action, these Z matrices appear in almost the same way as the background field matrices X, but they are absent from the gauge fixing term defining background field gauge. Thus, if we start with our expression for the effective action and integrate out further components of the fields K, we have essentially chosen a gauge different than the conventional background-field gauge for the 3-body system. As we have seen, with the standard choice of background-field gauge the one-loop matrix theory potential correctly reproduces supergravity interactions in a certain gauge. It may be rather nontrivial to extract the gravitational physics of the 3-body system if we choose an unorthodox gauge for the matrix theory calculation. (Issues along these lines were responsible for a discrepancy between calculations at the 2-loop level in [26, 27]).

These issues will probably need to be resolved to find a complete correspondence between the minimal 3-body interactions of matrix theory and supergravity.

4.4 Matrix theory in a general background

As a final application of our results, we discuss matrix theory in a general background. We would like to generalize the matrix theory action to one which includes a general supergravity background given by a metric tensor, 3-form field, and gravitino field which together satisfy the equations of motion of 11D supergravity. This issue has been discussed previously in [59, 60, 61, 62, 63, 64, 65]. In [63] it was argued that light-front M-theory on an arbitrary compact manifold should be reproduced by the low-energy 0-brane action on the same compact manifold; no explicit description of this low-energy theory was given, however. In [62] an explicit prescription was given for the first few terms of a matrix theory action on a general Kähler 3-fold which agreed with a general set of axioms proposed in [61]. In [59] and [64], however, it was argued that no finite N matrix theory action could correctly reproduce physics on a large K3 surface. We propose here an explicit formulation of matrix theory in an arbitrary background geometry. We leave an in-depth analysis of this proposal to further work and only sketch the basic structure of the theory here.

If we assume that matrix theory is a correct description of M-theory around a flat background, then there is a large class of curved backgrounds for which we know it is possible to construct a matrix theory action for $N \times N$ matrices. This is the class of backgrounds which can be produced as long-range fields produced by some other supergravity matter configuration with a known description in matrix theory. Imagine that a background metric $g_{IJ} = \eta_{IJ}+h_{IJ}$, a 3-form field A_{IJK} and a gravitino field ψ_I of light-front compactified 11-dimensional supergravity can be produced by a matter configuration described in matrix theory by matrices $\tilde{X}^a$. Then the matrix theory action describing $N \times N$ matrices X^a in this background should be precisely the effective action found by considering the block-diagonal matrix configuration

$$\mathbf{X}^a = \begin{bmatrix} X^a & 0 \\ 0 & \tilde{X}^a \end{bmatrix}$$

(and a similar fermion configuration) and integrating out the off-diagonal fields as well as fluctuations around the background $\tilde{X}$.

From the correspondence found in this paper and in [30], we know that for weak background fields, the first few terms in an expansion of this derived action in the background metric are given by

$$\begin{aligned} S_{\text{eff}} &= S_{\text{matrix}} + \int dx\; T^{IJ}(x) h_{IJ}(x) + \cdots \\ &= S_{\text{matrix}} + \int dx^+ \left\{ T^{IJ} h_{IJ}(0) + T^{IJ(i)} \partial_i h_{IJ}(0) + \cdots \right\} + \cdots , \end{aligned} \qquad (4.1)$$

where $T^{IJ(\cdots)}$ are the moments of the matrix theory stress-energy tensor, and there are analogous terms for the coupling of the membrane, 5-brane and fermionic components of the supercurrent to A_{IJK} and S_I. If the standard formulation of matrix theory in a flat background is correct, the absence of corrections to the long-range $1/r^7$ potential around an arbitrary matrix theory object up to at least order $1/r^{11}$ implies that this formulation must be correct at least up to terms of order $\partial^4 h$ and h^2. As we have derived it, this formulation of the effective action is only valid for certain background geometries which can be produced by well-defined matrix theory configurations. It is natural, however, to suppose that this result can be generalized to an arbitrary background. Thus, we propose that up to nonlinear terms in the background, the general form of the matrix theory action in an arbitrary but weak background is given by

$$S_{\text{weak}} = \int d\tau \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n} \frac{1}{n!} \Big(T^{IJ(i_1\cdots i_n)} \partial_{i_1} \cdots \partial_{i_n} h_{IJ} + J^{IJK(i_1\cdots i_n)} \partial_{i_1} \cdots \partial_{i_n} A_{IJK} + \\ + M^{IJKLMN(i_1\cdots i_n)} \partial_{i_1} \cdots \partial_{i_n} A^D_{IJKLMN} + \text{fermion terms} \Big) \qquad (4.2)$$

Let us make several comments about this action. First, this formulation is only appropriate for backgrounds with no explicit x^- dependence, as we do not understand how to encode higher modes in the compact direction in the components of the supergravity currents. Second, note that the coupling to A^D is free of ambiguity since the net 5-brane charge must vanish for any finite matrices, so that only first and higher derivatives of A^D appear in the action. Third, note that though we have only provided explicit expressions for the zeroeth and some of the first moments of T, the bosonic terms for the complete set of moments were given in [30], and we may in principle generalize the calculation of this paper to determine all the fermionic contributions from higher order terms in the one loop matrix theory potential.

The action (4.2) has precisely the form we would expect for the 11-dimensional supergravity action around a fixed classical background configuration. Typically, to study a quantum field theory in the presence of classical background fields, one replaces each field ϕ in the action with $\phi_{cl} + \phi'$ where ϕ_{cl} is the background field, generally taken to satisfy the equations of motion. Applying this presciption to supergravity with a background metric, we may expand the resulting action in powers of the background field (considering the background metric to be a small perturbation h_{IJ} away from a flat background). Because of the $\sqrt{g}$ in the gravity action, this will generate an infinite series of terms with arbitrary powers of h, which we may write as

$$S = S_{\text{SUGRA}} + \int dx \left\{ h_{IJ} T^{IJ} + h_{IJ} h_{KL} U^{IJKL} + \cdots \right\} , \qquad (4.3)$$

To leading order in h, this is the same as expression (3.1), but now h is the classical field and T^{IJ} is a quantum operator. Assuming that h depends only on the transverse coordinates, we may make a Taylor expansion about the origin, and write

$$\int dx\, h_{IJ}T^{IJ} = \int dx^{+} \left\{ h_{IJ}(0)T^{IJ} + \partial_i h_{IJ}(0)T^{IJ(i)} + \cdots \right\} , \tag{4.4}$$

where

$$T^{IJ(i_1 \ldots i_n)} \equiv \int dx^{-} d\vec{x} T^{IJ}(x) x^{i_1} \cdots x^{i_n} .$$

By generalizing this argument to include the 3-form and gravitino fields, we can derive the precise form of the action (4.1) which is linear in the background fields. Of course, it is desirable to generalize our proposal to a matrix theory formulation for the action which includes nonlinear terms in the background fields, corresponding to the higher order terms in (4.3). Thus we need matrix theory quantities corresponding to moments of the operator U in (4.3) and of the operators coupling to higher powers of h. However, in principle, we should be able to determine these from an analysis of higher-loop terms in the matrix theory potential for a two-body interaction. To see this, note that for the term U_m that couples to m powers of the background h, there should be a tree-level interaction in supergravity given by

$$\int dx_1 \ldots dx_m dz\, \hat{T}(x_1) \ldots \hat{T}(x_m) \tilde{U}_m(z) \langle h(x_1) \ldots h(x_m)(h(z))^m \rangle . \tag{4.5}$$

This will contain m propagators, and will give leading-order terms at order $1/r^{7m}$. On the matrix theory side this term must come from a diagram with at least m loops, since we need m separate traces to give the m factors of T here. But the m loop contribution in matrix theory is expected to vanish at orders less than $1/r^{7m}$ (for agreement with classical supergravity), so we see that the interaction (4.5) must appear in the leading-order m-loop terms. From these terms it should be possible to read off U_m as the tilded quantity coupled to m hatted Ts.

We have now given an explicit formulation of the matrix theory action in an arbitrary background. The bosonic terms in the action are completely determined to linear order in the background by the higher moments of the supergravity currents given in [30]. The terms containing fermions are determined to quadratic order in the fermions to first order in the derivatives of the backgrounds by the results in this paper. In general, the terms appearing at mth order in the backgrounds can be precisely determined by a m-loop matrix theory calculation. If matrix theory is correct in a flat space-time background, then this formulation must be correct, at least for backgrounds which can be generated by other matrix theory configurations. We hypothesize that the proposal is correct in a more general class of backgrounds; more work is necessary, however, to determine the validity of this generalization of the proposal. We conclude this section with a few brief comments on this proposed action.

Our derivation of the general background matrix theory action has been completely within the context of matrix theory, however in priciple one should be able

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to follow the arguments of [63] to derive this action as a limit of the action for D0 branes in an arbitrary type IIA supergravity background. It turns out that the necessary terms in the D0 brane action are known only for the case $N = 1$. In this case, the matrix theory action derived directly from the single D0 brane action exactly matches our result here. Requiring that such a derivation should work for the case $N > 1$, one may use our result to derive previously unknown leading terms in the action for multiple D0 branes in an arbitrary type IIA supergrvity background. These results will be presented in a forthcoming paper [69].

One other approach which one might take to define matrix theory in a general background geometry is to follow the original derivation of matrix theory as a regularized membrane theory, but to include a general background geometry instead of a flat background as was used in [4, 6]. The superspace formulation of a supermembrane theory in a general 11D supergravity background was given in [5]. In principle, it should be possible to simply apply the matrix regularization procedure to this theory to derive matrix theory in a general background geometry. Unfortunately, however, the connection between superspace fields and component fields is not well-understood in this theory. Until recently, in fact, the explicit expressions for the superspace fields were only known up to first order in the component fermion fields θ [66]. In [67], this analysis was extended to quadratic order in θ with the goal of finding an explicit formulation of the supermembrane in general backgrounds in terms of component fields, to which the matrix regulation procedure could be applied to generate a general background formulation of matrix theory. It would be very interesting to carry this work through and compare with our proposal (4.2), (4.3) for the general background matrix theory action.

It was pointed out by Douglas in [61] that a natural Ansatz for matrix theory in a curved background metric can be made by simply covariantizing the action in a flat background. For the bosonic terms, this would lead to an action of the form

$$S_{\text{covariant}} = \int d\tau \ \mathrm{Tr} \ \left(g_{ij}(X) D_\tau X^i D_\tau X^i + g_{ij}(X) g_{kl}(X) [X^i, X^k][X^j, X^l] \right) . \qquad (4.6)$$

Douglas suggested that the correct action for matrix theory in a general background should be a supersymmetrized version of this action in which the ordering ambiguities inherent in the traces are somehow resolved. In fact, our proposal (4.2) corresponds precisely to this sort of structure. It is straightforward to check that the terms which arise in an expansion of (4.6) correspond precisely to the components described in (3.16) for the components of the stress-energy tensor coupling to h_{IJ}. The ordering ambiguities are resolved by using the explicit formulae we have derived for the higher moments of the stress tensor. In a general background, the theory is no longer in fact supersymmetric, since an arbitrary background metric can break the supersymmetry. Nonetheless, the fermionic contributions to the action arise from the generalizations of the expressions we have given for the fermionic contributions to the supergravity currents. We note also that the argument which suggested (4.6) is not easily generalized to suggest a form for the couplings of the membrane and 5-brane currents to a background 3-form field, or for the couplings of the fermionic current components to a background gravitino field.

In [61], it was proposed that any formulation of matrix theory in a curved background should satisfy a number of axioms. All these axioms are satisfied in a straightforward fashion by our proposal, except one: this exception is the axiom that states that a pair of 0-branes at points x^i and y^i should correspond to diagonal 2×2 matrices where the masses of the off-diagonal fields should be equal to the geodesic distance between the points x^i and y^i in the given background metric. In [69] we verify this axiom for the linear terms in the background h which we have described here. It may be possible to find situations in which this axiom is violated for our proposal at order h^2 or higher. We note, however, that in order to demonstrate the correspondence between linearized supergravity and matrix theory interactions it was necessary to choose a gauge for the graviton field h_{IJ}. The explicit formulae given here and in [30] for the supergravity currents depended explicitly on this choice of gauge. Thus, if we use these expressions for the currents in (4.2) we should expect that the background must be chosen in the same gauge that we have chosen to fix the graviton propagator.

Thus, we synopsize the discussion in this section as follows: we have given a proposal for the matrix theory action in a general metric, 3-form and gravitino background. This formulation of the action requires the explicit computation of matrix expressions for higher moments of all the supergravity currents. At linear order in the background fields, these expressions can all be found from a one-loop matrix theory computation, although we have only performed the analysis explicitly to quadratic order in fermion backgrounds so far. At mth order in the background fields, matrix expressions are needed for quantities which can be determined from a m-loop matrix theory calculation. The definitions of the matrix theory currents depend upon gauge choices for the propagating supergravity fields. For a given gauge choice, the theory is only defined for backgrounds compatible with the gauge condition. We leave the further study of this proposal for matrix theory in a general background to further work. It would clearly be interesting to understand how the issues raised in [59, 64] can be understood in the context of the action we have proposed. It would also be interesting to consider this action on a general Kähler manifold and to understand the connection with the results of [62]. Finally, it would be interesting to study this proposal for general backgrounds in the T-dual context of the matrix string theory formulation of Dijkgraaf, Verlinde and Verlinde [68]. In that framework, our proposal gives a description of a fundamental string theory in a general background which can have spatially dependent Ramond-Ramond fields.

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A. The calculation in detail

In this section, we present in detail our calculation of the one-loop matrix theory effective action for a background corresponding to widely separated systems. The setup was presented in section 2.1 above, along with the action and propagators.

We divide the calculation up by the number of fermionic fields appearing in the result. The leading term with $2n$ fermions comes from choosing n each of the vertices (2.5) and (2.6). Such a term will have n bosonic and n fermionic propagators, and thus appear at order $3n-1$. Thus, up to order $1/r^7$, where we expect the leading-order interactions, we will have terms with zero, two, and four fermions.

A.1 Bosonic terms

For this part of the calculation, we set $K_0 = 0$, and calculate derivative terms explicitly. We have purely bosonic terms coming from a boson loop, a fermion loop, and a ghost loop mentioned before.

For $K_0 = 0$, the bosonic vertices (2.7) and (2.8) vanish, so the complete contribution from the bosonic loop may be written as a sum over the number of insertions of the bosonic vertex (2.9) as

$$\begin{aligned}\Gamma^{\text{bosloop}} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int \prod d\tau_i \operatorname{Tr} \langle (Y^\dagger_{a_1}(\tau_1) M_{a_1 b_1}(\tau_1) Y_{b_1}(\tau_1)) \cdots \\ &\qquad\qquad \cdots (Y^\dagger_{a_n}(\tau_n) M_{a_n b_n}(\tau_n) Y_{b_n}(\tau_n)) \rangle \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int \prod d\tau_i \operatorname{Tr}\, (M_{a_1 a_2}(\tau_1) M_{a_2 a_3}(\tau_2) \cdots M_{a_n a_1}(\tau_n)) \times \\ &\qquad\qquad \times \Delta(\tau_1 - \tau_2)\Delta(\tau_2 - \tau_3) \cdots \Delta(\tau_n - \tau_1)\,.\end{aligned}$$

In order to get the complete set of derivative and non-derivative terms in the effective action, we now perform a Taylor expansion of all functions about a specific time, which we choose to be τ_1 and perform the remaining τ integrals. Defining $\sigma_i = \tau_i - \tau_1$ and $\tau = \tau_1$ we have

$$\Gamma^{\text{bosloop}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{D_i=0}^{\infty} \int d\tau \operatorname{Tr} \left(M_{a_1 a_2}(\tau) M^{(D_2)}_{a_2 a_3}(\tau) \cdots M^{(D_n)}_{a_n a_1}(\tau) \right) \times$$

$$
\begin{aligned}
&\times \int \prod_{i=1}^{n} \frac{dk_i}{2\pi} \int \prod_{i=2}^{n} \left(\frac{d\sigma_i \sigma_i^{D_i}}{D_i!} \right) \frac{e^{-ik_1\sigma_2}}{k_1^2 + r^2} \frac{e^{ik_2(\sigma_2-\sigma_3)}}{k_2^2 + r^2} \cdots \frac{e^{ik_n\sigma_n}}{k_n^2 + r^2} \\
=& \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{D_i=0}^{\infty} \prod \frac{1}{D_i!} \int d\tau \mathrm{Tr} \left(M_{a_1a_2}(\tau) M_{a_2a_3}^{(D_2)}(\tau) \cdots M_{a_na_1}^{(D_n)}(\tau) \right) \times \\
&\times \int \frac{dk_1}{2\pi} \left(\prod_{i=2}^{n} dk_i \{ (-i\partial_{k_i})^{D_i} \delta(k_i - k_{i-1}) \} \right) \frac{1}{k_1^2 + r^2} \cdots \frac{1}{k_n^2 + r^2} \\
=& \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{D_i=0}^{\infty} \prod \frac{1}{D_i!} \int d\tau \mathrm{Tr} \left(M_{a_1a_2}(\tau) M_{a_2a_3}^{(D_2)}(\tau) \cdots M_{a_na_1}^{(D_n)}(\tau) \right) \times \\
&\times \frac{1}{r^{2n+D-1}} \int \frac{dk}{2\pi} \frac{1}{k^2+1} (i\partial_k)^{D_2} \left\{ \frac{1}{k^2+1} \cdots (i\partial_k)^{D_n} \left\{ \frac{1}{k^2+1} \right\} \cdots \right\}.
\end{aligned}
$$

The k integral is a sum of integrals of the type

$$
\int \frac{k^n}{(1+k^2)^m} ,
$$

which are always convergent (for $n < 2m - 1$ and all cases which may arise here) and easily computed. Note that this integral vanishes for odd n, so we will only have terms with an even number of derivatives. In an exactly analogous fashion, the fermion loop contribution can be computed to be

$$
\begin{aligned}
\Gamma^{\text{fermiloop}} =& - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{D_i=0}^{\infty} \prod \frac{1}{D_i!} \int d\tau \mathrm{Tr} \left(K_{i_1}(\tau) K_{i_2}^{(D_2)}(\tau) \cdots K_{i_m}^{(D_m)}(\tau) \right) \times \\
&\times \int \frac{dk}{2\pi} \mathrm{Tr} \left(\gamma^{i_1} \frac{\not{r} + ik}{k^2 + r^2} \gamma^{i_2} (i\partial_k)^{D_2} \left\{ \frac{\not{r} + ik}{k^2 + r^2} \cdots \gamma^{i_m} (i\partial_k)^{D_n} \left\{ \frac{\not{r} + ik}{k^2 + r^2} \right\} \right\} \right).
\end{aligned}
$$

Up to order $1/r^8$we find again that only terms with an even number of derivatives contribute, since the other terms either have an odd number of k's in the numerator or an odd number (less than nine) of gamma matrices in the trace.

Finally, the ghost loop contributes a term

$$
\begin{aligned}
-2 \sum_{n=1, D_i=0}^{\infty} & \frac{(-1)^n}{n D_i!} \int d\tau \mathrm{Tr} \Bigg((K^2 + 2r \cdot K)(\tau) \times \\
&\times (K^2 + 2r \cdot K)^{(D_2)}(\tau) \cdots (K^2 + 2r \cdot K)^{(D_n)}(\tau) \Bigg) \times \\
&\times \frac{1}{r^{2n+D-1}} \int \frac{dk}{2\pi} \frac{1}{k^2 + 1} (i\partial_k)^{D_2} \left\{ \frac{1}{k^2 + 1} \cdots (i\partial_k)^{D_n} \left\{ \frac{1}{k^2 + 1} \right\} \cdots \right\}.
\end{aligned}
$$

Using these general results, we can explicitly calculate the action at each order in $1/r$, (with the help of Maple) and we find that all terms with explicit derivatives cancel up to and including order $1/r^7$, leaving exactly the action

$$
\Gamma_{1/r^7}^{\text{bos}} = \frac{15}{16} \mathrm{STr} \left(F_{ab} F_{bc} F_{cd} F_{da} - \frac{1}{4} F_{ab} F_{ab} F_{cd} F_{cd} \right) ,
$$

calculated in [38] using the quasistatic approximation. Thus, to this order, the result of our full calculation agrees with the quasistatic calculation. However, as we shall see below, we expect the appearance of higher derivative terms at order $1/r^9$ which would not appear in a quasistatic approximation.

We will also be interested in terms at $1/r^8$. At this order, we find two contributions. The first is proportional to an insertion of $r \cdot K$ into the $1/r^7$ action,

$$-\frac{105}{16}\mathrm{STr}\,\left(F_{ab}F_{bc}F_{cd}F_{da}(r\cdot K)-\frac{1}{4}F_{ab}F_{ab}F_{cd}F_{cd}(r\cdot K)\right).$$

The second contribution contains a nine index totally antisymmetric tensor coming from the trace of nine gamma matrices in a fermion loop term, and is given by

$$\frac{35r_s}{256r^9}\mathrm{STr}\,\left(F_{ij}F_{kl}F_{mn}\dot{K}_pK_q\right)\epsilon^{ijklmnpqs}.$$

A.2 Two-fermion terms

We now move on to terms with two fermions. These arise from loops with a single insertion of each of the two boson-fermion vertices (2.6) and (2.5) to give two factors of L. In addition, we may have an arbitrary number of boson-boson vertices and fermion-fermion vertices. In this case, we shall proceed with the calculation using a non vanishing K_0 and ignoring derivative terms. In this case, we have two types of bosonic vertices,

$$-Y_a^\dagger M_{ab}Y_b\,,$$

which we had before, plus the additional vertex

$$2iY_a^\dagger K_0\partial_\tau Y_a\,.$$

The derivative here gives an extra factor of k when acting on a propagator. Following the same steps as in the calculation above (but keeping only the first term in the Taylor expansions) we find the two fermion term to be

$$\begin{aligned}\Gamma^{\mathrm{LL}} = &\sum_{m,n,p=0}^{\infty}(-1)^{m+n+p}2^p\int d\tau\,\mathrm{Tr}\left(L_\alpha K_{a_1}\cdots K_{a_m}L_\beta\,\times\right.\\ &\times\sum_{K_0\,insertions}(M_{b_1b_2}\cdots M_{b_nb_{n+1}};K_0^p))\,\times\\ &\times\int\frac{dk}{2\pi}\left(\gamma^{b_{n+1}}\frac{\not{\eta}+ik}{k^2+r^2}\gamma^{a_1}\frac{\not{\eta}+ik}{k^2+r^2}\cdots\gamma^{a_m}\frac{\not{\eta}+ik}{k^2+r^2}\gamma^{b_1}\frac{k^p}{(k^2+r^2)^{n+p+1}}\right)_{\alpha\beta}\end{aligned}\tag{A.1}$$

Here, we sum over the $\binom{n+p+1}{n}$ distinct ways of ordering the p K_0's and m M's.

Simplification of this expression becomes quite difficult even at the orders we are interested in. Though it is possible to calculate the integrals by computer, and perhaps to display all terms with gamma matrices rearranged into some standard form, the number of distinct terms calculated blindly in this way would be on the order of 500 at order $1/r^7$, making it a virtually impossible task to simplify into a meaningful expression. Thus, at this point, it is useful to make a couple of observations which will simplify the calculation considerably.

First, since our answer must have the SO(9) rotational symmetry of the original action, any r_i must appear in an r^2, an $r \cdot K$, or coupled to a gamma matrix. For two-fermion terms, all gamma matrices appear together in a product between the L's, so we may always write such terms with a maximum of one r_i coupled to a gamma matrix (since $\not{r}\not{r} = r^2$). Now, note that in the action (2.2), r_i and K_i appear only in the combination $(r_i + K_i)$ (a trivial consequence of their definitions). Thus, a transformation

$$r_i \to r_i + \Lambda_i \tag{A.2}$$

is exactly equivalent to a transformation

$$K_i \to K_i + \Lambda_i\,. \tag{A.3}$$

Applying transformation (A.2) to the effective action we see that for each term containing some number of $r \cdot K$'s and/or an r_i coupled to a gamma matrix, we would get a single term in which all of these r's, but no others, were replaced with Λs. Such a term must also appear when we apply the equivalent transformation (A.3) to the original effective action, and we see that it can come only from a term in which all r's appear only as r^2's. We conclude that we may ignore all terms in our calculation containing $r \cdot K$ or a single r coupled to a gamma matrix, restoring them at the end by making a transformation (A.3) on the action, then replacing all Λs not appearing as Λ^2 with rs, and setting Λ to zero. The only caveat is that one must calculate to higher orders to find the terms of a given order which contain $r \cdot K$ or r_i coupled to a gamma matrix, however, we will often be able to deduce the form of the relevant higher order terms without doing the full calculation.

Consider first the terms in (A.2) which contain only spatial Ks. For these terms, $p = 0$ and all the indices a_i are spatial. By our simplifying assumption above, we may take $\not{r}$ to anticommute with any γ^i whose index is coupled to a K, since the anticommutator produces a term containing $r \cdot K$ which we ignore. Thus, effectively, we have the relation

$$\frac{\not{r} + ik}{k^2 + r^2}\gamma^i \frac{\not{r} + ik}{k^2 + r^2} = -\frac{\gamma^i}{r^2 + k^2}\,.$$

Using this, the general form (A.2) simplifies considerably. For even orders in $1/r$ (even m), we are left with a single $(\not{r} + ik)$ in the numerator, and we ignore all such terms, since they either contain a single r coupled to a gamma matrix or a vanishing integral of $k/(r^2+k^2)^n$. Thus we may ignore all K_0 independent terms at even orders in $1/r$. For odd orders in $1/r$ the integral in (A.2) becomes simply

$$\int \frac{dk}{2\pi} \frac{(-1)^{(m+1)/2}}{(r^2+k^2)^{n+m/2+3/2}} = \frac{(2n+m)!!}{(n+m/2+1/2)!2^{(n+m/2+3/2)} r^{2n+m+2}}\,,$$

so the K_0 independent terms can be written

$$\Gamma^{\mathrm{LL}} = \sum_{l=0}^{\infty} \sum_{m+2n=2l-1} \frac{a_l}{r^{2l+1}} \int d\tau \ \mathrm{Tr}\,(L_\alpha K_{i_1} \cdots K_{i_m} L_\beta M_{b_1 b_2} \cdots M_{b_n b_{n+1}}) \times \\ \times (\gamma^{b_n+1} \gamma^{i_1} \cdots \gamma^{i_m} \gamma^{b_1})_{\alpha\beta}\,, \tag{A.4}$$

where we have the same coefficient for all terms of a given order, given by

$$a_l = \frac{(-1)^{l-1}}{2^{2l+1}} \binom{2l}{l}\,.$$

An analogous (but simpler) chain of reasoning shows that the $0+0$ action (2.10) gives exactly the same expression for the terms involving only spatial Ks, but with a different a_l. Though we do not necessarily expect this equivalence to hold between the full $0+0$ and $0+1$ dimensional calculations (the $0+0$ dimensional action displays an SO(10) symmetry while the $0+1$ dimensional calculation does not), it is interesting and helpful to carry out the simpler $0+0$ dimensional calculation including the K_0 terms. We will see that this calculation suggests the eventual form of the $0+1$ dimensional effective action.

From the action (2.10), we see that the complete set of two fermion terms in the $0+0$ dimensional calculation at order $1/r^q$ is proportional to:

$$\sum_{m+2n+2=q} \frac{1}{r^{2n+2m+3}} \mathrm{Tr}\,(L_\alpha K_{a_1} \cdots K_{a_m} L_\beta M_{b_1 b_2} \cdots M_{b_n b_{n+1}}) (\gamma^{b_n+1} \not{r} \gamma^{a_1} \not{r} \cdots \gamma^{a_m} \not{r} \gamma^{b_1})_{\alpha\beta} \tag{A.5}$$

Unfortunately, we can no longer assume that $\not{r}$ anticommutes with the other γ matrices, since the indices a_i may now take a value of 0, in which case $\not{r}$ and γ^{a_i} clearly commute. However, defining γ_a (lower index) such that

$$\gamma_i = \gamma^i\,, \qquad \gamma_0 = -\gamma^0 = i\,,$$

(index lowered by η, not δ!) we note that

$$\gamma^a \gamma_b + \gamma^b \gamma_a = 2\delta_{ab}$$

and

$$\gamma_a \gamma^b + \gamma_b \gamma^a = 2\delta_{ab}\,.$$

Finally, since $r_0 = 0$, we have $r_a \gamma^a = r_a \gamma_a$, and we find

$$\begin{aligned} \not{r} \gamma^{a_1} \not{r} \cdots \gamma^{a_m} \not{r} &= \gamma_{b_1} \gamma^{a_1} \gamma_{b_2} \cdots \gamma^{a_m} \gamma_{b_{m+1}} r_{b_1} \cdots r_{b_{m+1}} \\ &= \begin{cases} \gamma_{a_1} \gamma^{a_2} \gamma_{a_3} \cdots \gamma^{a_{m-1}} \gamma_{a_m} r^{m+1} & (m \text{ odd}) \\ \gamma_{a_1} \gamma^{a_2} \gamma_{a_3} \cdots \gamma^{a_m} \gamma_b r_b r^m & (m \text{ even})\,. \end{cases} \end{aligned} \tag{A.6}$$

We may choose to ignore the even m terms since they have a single r coupled with a gamma matrix, but in either case, we see that there is an odd number of gamma matrices with alternating upper and lower indices. This suggests an upgrade to ten dimensional notation as follows.

First, note that the 32×32 matrices defined by:

$$\Gamma^a = \begin{bmatrix} 0 & \gamma^a \\ \gamma_a & 0 \end{bmatrix}$$

satisfy the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$$

and thus are true gamma matrices for an SO(10) spinor representation. If we define a 32 component spinor by

$$\mathbf{L} = \begin{pmatrix} 0 \\ L \end{pmatrix}$$

(ie satisfying a Weyl condition $\Gamma^{10}\mathbf{L} = \mathbf{L}$) we find that

$$\begin{aligned} \bar{\mathbf{L}}\Gamma^a\mathbf{L} &= iL\gamma^a L \\ \bar{\mathbf{L}}\Gamma^a\Gamma^b\Gamma^c\mathbf{L} &= iL\gamma^a\gamma^b\gamma^c L \\ \bar{\mathbf{L}}\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e\mathbf{L} &= iL\gamma^a\gamma_b\gamma^c\gamma_d\gamma^e L \end{aligned} \tag{A.7}$$

and so on, while expressions with an even number of Γs sandwiched between $\bar{\mathbf{L}} = \mathbf{L}\Gamma^0$ and $\mathbf{L}$ vanish. Thus, we see that the alternating upper and lower indices in (A.6) may be recovered using an ordinary product of $D = 10$ gamma matrices. From now on we will drop the bold font on 32 component spinors, as the dimension should be clear from the context.

We may therefore rewrite the expression (A.5) as

$$\sum_{m+2n+2=q \; odd} \frac{1}{r^{2n+m+2}} \mathrm{Tr}\,(\bar{L}_\alpha K_{a_1} \cdots K_{a_m} L_\beta M_{b_1 b_2} \cdots M_{b_n b_{n+1}})(\Gamma^{b_n+1}\Gamma^{a_1} \cdots \Gamma^{a_m}\Gamma^{b_1})_{\alpha\beta}\,, \tag{A.8}$$

where as above, we may ignore the terms at even orders in $1/r$. But this is exactly obtained from the $1/r^q$ term in (A.4) by replacing γ^a with Γ^a everywhere and the left L with $\bar{L}$. Since manipulations with γ matrices and with Γ matrices are formally identical, we may actually get the full set of terms from the K_0 independent terms in this way regardless of what form the K_0 independent terms are in.

Using the tools just outlined, we would like to write the expression (A.2) for the two-fermion terms in a more enlightening form. In particular, we will see that at each order in $1/r$, we can divide the terms into a set which are the dimensional reduction of $D = 10$ Lorentz and gauge invariant terms, plus some remaining terms which do not display the full $D = 10$ symmetry. These terms must still respect the symmetries of the Matrix theory action, namely $0+1$ dimensional gauge invariance and SO(9) rotational invariance in the (formerly) spatial indices. We proceed order by order in $1/r$, and in each case we begin by computing the $0+0$ dimensional result and then determine the remaining terms in the $0+1$ dimensional action.

A.2.1 Order $1/r^3$

Beginning at order $1/r^3$, we find a single term in expression (A.8), taking $m = 1, n = 0$, proportional to

$$\mathrm{Tr}\,(\bar{L}K_a\Gamma^b\Gamma^a\Gamma^b L) \propto -4\mathrm{Tr}\,(\bar{L}\Gamma^a[K_a, L])\,.$$

This is the dimensional reduction to zero dimensions of a ten dimensional Lorentz and gauge invariant term proportional to

$$\mathrm{Tr}\,(\bar{L}\not{D}L)\,.$$

Now, from the general $0+1$ dimensional expression (A.2) we see that at $1/r^3$, we have terms coming from $(p = 0, n = 0, m = 1)$ and $(p = 1, n = 0, n = 0)$. These are easily calculated, and combine to give the same expression (up to the overall coefficient) as in the 0+0 calculation,

$$\begin{aligned}
\Gamma^{0+1}_{1/r^3} &= i\mathrm{Tr}\,(\bar{L}\Gamma^a[K_a, L]) \\
&= i\mathrm{Tr}\,(\bar{L}\Gamma^i[K_i, L]) + i\mathrm{Tr}\,(\bar{L}\Gamma^0[K_0, L]) \\
&= i\mathrm{Tr}\,(\bar{L}\Gamma^i[K_i, L]) + \mathrm{Tr}\,(\bar{L}\Gamma^0 D_0 L)\,,
\end{aligned}$$

where in the last line we have restored the derivative term by replacing the K_0 commutator by a covariant derivative (dictated by $0+1$ dimensional gauge invariance). We see that this is the dimensional reduction to $0+1$ dimensions of the same expression,

$$\Gamma_{1/r^3} = \mathrm{Tr}\,(\bar{L}\not{D}L)$$

that gave the $0+0$ dimensional action. Hence, we may guess that for any terms in the $0+0$ dimensional action which are the dimensional reduction of $D = 10$ Lorentz and gauge covariant terms, there will be terms in the $0+1$ dimensional action which are the dimensional reduction of the same terms. The coefficient of these is determined by the expression (A.4). The presence of nonvanishing terms in the effective action below order $1/r^7$ may seem surprising in light of the supposed correspondence with supergravity, whose leading-order interactions are at $1/r^7$. However, to find a potential between two physical systems, we need to require that the matrix theory equations of motion be satisfied by the background, and we shall see that this results in cancellation for all terms below order $1/r^7$.

A.2.2 Order $1/r^5$

At order $1/r^5$, the $0+0$ dimensional expression (A.8) gives terms proportional to

$$\frac{-3}{16}\Big(\mathrm{Tr}\,(\bar{L}_\alpha K_{a_1}K_{a_2}K_{a_3}L_\beta)(\Gamma^b\Gamma^{a_1}\Gamma^{a_2}\Gamma^{a_3}\Gamma^b)_{\alpha\beta}+\mathrm{Tr}\,(\bar{L}_\alpha K_a L_\beta M_{b_1b_2})(\Gamma^{b_2}\Gamma^a\Gamma^{b_1})_{\alpha\beta}\Big)\,.$$

We have chosen coefficients using (A.4) so that K_0 independent terms agree with those in the $D=0+1$ calculation.

Manipulating these expressions, we find terms which are the dimensional reduction of the $D=10$ covariant action

$$\begin{aligned}\Gamma^{\mathrm{cov}}_{1/r^5}=&\frac{3}{8}i\,\mathrm{Tr}\,(\bar{L}\,\Gamma^b\,D_aF_{ab}\,L)+\\&+\frac{3}{16}i\,\mathrm{Tr}\,(\bar{L}\,\Gamma^{[ab]}\,F_{ab}\,\not{D}L)-\frac{3}{16}i\,\mathrm{Tr}\,(\bar{L}\,\Gamma^{[ab]}\,\not{D}L\,F_{ab})\end{aligned}\tag{A.9}$$

plus additional terms

$$\frac{3}{4}i\Big(\mathrm{Tr}\,(\bar{L}K^2[\not{K},L])+\mathrm{Tr}\,(\bar{L}[\not{K},L]K^2)\Big)\,.\tag{A.10}$$

At this point, it is convenient to define an operation $\mathrm{Sym}\,(\mathrm{Tr}\,(A_1\ldots A_n);B_1;\ldots,B_n)$ to be the average of all possible different insertions of the operators B_i between elements of the trace. For example,

$$\begin{aligned}\mathrm{Sym}(\mathrm{Tr}\,(\bar{L}\not{D}L);\vec{K}^2,D_0^2)=&\frac{1}{6}(\mathrm{Tr}\,(D_0^2\bar{L}\vec{K}^2\,\not{D}L)+\mathrm{Tr}\,(\bar{L}\,D_0^2\vec{K}^2\,\not{D}L)+\mathrm{Tr}\,(\bar{L}\,\vec{K}^2\,D_0^2\not{D}L)+\\&+\mathrm{Tr}\,(D_0^2\bar{L}\,\not{D}L\,\vec{K}^2)+\mathrm{Tr}\,(\bar{L}\,D_0^2\not{D}L\,\vec{K}^2)+\mathrm{Tr}\,(\bar{L}\,\not{D}L\,D_0^2\vec{K}^2)).\end{aligned}$$

Note that any commutators in the trace are treated as single element.

Using this definition, the eqpression (A.10) may be written

$$\frac{3}{2}i\,\mathrm{Sym}\,(Tr(\bar{L}[\not{K},L]);K^2)\,.$$

Thus, we see that the non-covariant term is proportional to the average of all possible insertions of K^2 into the covariant $1/r^3$ action. Now, K^2 is not gauge invariant in the $D=0+1$ sense since it contains K_0^2, so we must have additional terms in the $D=0+1$ action which restore gauge invariance. We know, however, that the $D=0+1$ and $D=0+0$ actions contain the same K_0 independent terms, so we expect a term

$$\frac{3}{2}i\,\mathrm{Sym}\,(Tr(\bar{L}\not{D}L);\vec{K}^2)\tag{A.11}$$

in the $D=0+1$ action plus additional terms whose sum with

$$\frac{3}{2}i\,\mathrm{Sym}\,(Tr(\bar{L}\not{D}L);K_0^2)\tag{A.12}$$

is gauge invariant. Using (A.2) we may calculate all terms in the $D = 0+1$ action at $1/r^5$, and subtracting off a set of terms corresponding to (A.9), (A.11) and (A.12), it is not hard to show that the remaining terms are

$$-\frac{1}{2}i\ \mathrm{Sym}\left(Tr(\bar{L}\not{D}L); K_0, K_0\right). \qquad (A.13)$$

As desired, these combine with the term (A.12) to give a $0+1$ dimensional gauge invariant term

$$\frac{1}{4}i\mathrm{Tr}\left(\bar{L}[K_0, [K_0, \not{D}L]\right),$$

which we write (reinstating the covariant derivatives) as

$$\frac{1}{4}i\ \mathrm{Sym}\left(Tr(\bar{L}\not{D}L); D_0^2\right),$$

an average over the insertions of the covariant derivative squared into the covariant action at the previous order.

A.2.3 Order $1/r^7$

At order $1/r^7$, the picture is not significantly different. The $D = 0+0$ calculation gives terms which are the dimensional reduction of the $D = 10$ covariant action

$$\begin{aligned}
\Gamma^{\mathrm{cov}}_{1/r^7} = & -\frac{5}{64}\mathrm{Tr}\left(\bar{L}\ \Gamma^e\Gamma^a\Gamma^b\Gamma^d\ F_{ab}\,\not{D}L\,F_{de}\right) - \\
& -\frac{5}{16}\mathrm{Tr}\left(\bar{L}\ \Gamma^a\Gamma^c\ F_{ab}\,F_{bc}\,\not{D}L\right) - \frac{5}{16}\mathrm{Tr}\left(\bar{L}\ \Gamma^c\Gamma^a\ \not{D}L\,F_{ab}\,F_{bc}+\right) \\
& +\frac{5}{32}\mathrm{Tr}\left(\bar{L}\ \Gamma^d\Gamma^b\Gamma^c\ D_aF_{ab}\,L\,F_{cd}\right) - \\
& -\frac{5}{16}\mathrm{Tr}\left(\bar{L}\ \Gamma^c\ D_aF_{ab}\,L\,F_{cb}\right) - \frac{5}{16}\mathrm{Tr}\left(\bar{L}\ \Gamma^c\ F_{cb}\,D_aF_{ab}\,L\right) + \\
& +\frac{15}{16}\mathrm{STr}\left(\bar{L}\ \Gamma^b\Gamma^c\Gamma^d\ F_{ab}\,F_{cd}\,D_aL\right)
\end{aligned}$$

plus additional terms proportional to all possible insertions of K^2 into the covariant terms at $1/r^5$, and all insertions of two K^2 terms into the $1/r^3$ action. The $D = 0+1$ action contains additional terms proportional to the sum of all possible insertions of two K_0's into terms in the $1/r^5$ action, and as in the $1/r^5$ case these combine with terms containing insertions of K_0^2 to give terms which correspond to all possible insertions of D_0^2 into the $1/r^5$ terms. We will write the full $1/r^7$ action with coefficients below.

A.2.4 Even orders, $r \cdot K$ terms

Recalling the prescription above for restoring terms with r_i not appearing in r^2, we can take the terms calculated so far, make a transformation $K_i \to K_i + \Lambda_i$, replace

Λ with r anywhere it is coupled to K or a gamma matrix, and set Λ to zero. Noting that any covariant term is invariant under such a transformation, we see that the only new terms will be those obtained by replacing any number of $\vec{K}^2$s in a given term by $2r \cdot K$.

We may also have $r \cdot K$ terms coming by our prescription from higher order terms. However, it is easy to show that all $r \cdot K$ terms are simply insertions of $r \cdot K$ into lower order terms: By the equivalence of transformations (A.2) and (A.3), for any term containing $r \cdot K$ we must have a lower order term given by the removal of $r \cdot K$, since (A.3) produces a term of this form, and (A.2) leaves the form of all terms unchanged.

The only other possible term we may have neglected would be a term at even order containing a single r_i coupled to a gamma matrix. However, such a term would have to come from a term at one higher order with a gamma matrix coupled to a K_i not appearing in a commutator. Since no such terms appear at orders up to $\frac{1}{r^7}$, the only odd order terms appearing up to order $1/r^6$ will be given by insertions of odd numbers of $r \cdot K$s into terms calculated above.

A.2.5 Order $1/r^8$

In the applications below, it will be useful to determine some of the terms at order $\frac{1}{r^8}$. Using the $0+0$ dimensional action, we find that apart from terms containing $r \cdot K$, the two-fermion terms at $1/r^8$ are given by the $D = 10$ gauge invariant expression

$$\begin{aligned}\Gamma_{1/r^8} = & -\frac{35}{256 r^9} r_g \mathrm{STr}\left(\bar{L}\, F_{ab} F_{cd} F_{ef} \Gamma^{[abcdefg]} L\right) - \frac{105}{32 r^9} r_e \mathrm{STr}\left(\bar{L}\, F_{ab} F_{bc} F_{cd} \Gamma^{[ade]} L\right) - \\ & -\frac{105}{128 r^9} r_e \mathrm{STr}\left(\bar{L}\, F_{ab} F_{ab} F_{cd} \Gamma^{[cde]} L\right), \end{aligned} \tag{A.14}$$

which should also be correct for 0+1 dimensions. Note that in the prescription above, these terms would have come from non-covariant terms at $1/r^9$ with a gamma matrix coupled to a single free K_i not appearing in a commutator. No such terms appeared at lower odd orders, thus we did not have any terms of the form (2.19) at lower even orders.

A.2.6 Off-shell action to order $1/r^7$

Collecting the various terms calculated above, we now write down the complete off-shell action to order $1/r^7$. The terms which come from the dimensional reduction of a $D = 10$ Lorentz and gauge invariant action are given by

$$\begin{aligned}\Gamma^{\mathrm{cov}}_{1/r^3} &= \mathrm{Tr}\,(\bar{L} \not{D} L) \\ \Gamma^{\mathrm{cov}}_{1/r^5} &= \frac{3}{8} i\, \mathrm{Tr}\,(\bar{L}\, \Gamma^b\, D_a F_{ab}\, L) + \\ & + \frac{3}{16} i\, \mathrm{Tr}\,(\bar{L}\, \Gamma^{[ab]}\, F_{ab} \not{D} L) - \frac{3}{16} i\, \mathrm{Tr}\,(\bar{L}\, \Gamma^{[ab]} \not{D} L\, F_{ab})\end{aligned}$$

$$
\begin{aligned}
\Gamma^{\rm cov}_{1/r^7} = &-\frac{5}{64}{\rm Tr}\,(\bar{L}\,\Gamma^e\Gamma^a\Gamma^b\Gamma^d\;F_{ab}\,\not{D}L\,F_{de}) - \\
&-\frac{5}{16}{\rm Tr}\,(\bar{L}\,\Gamma^a\Gamma^c\;F_{ab}\,F_{bc}\,\not{D}L) - \frac{5}{16}{\rm Tr}\,(\bar{L}\,\Gamma^c\Gamma^a\;\not{D}L\,F_{ab}\,F_{bc}) + \\
&+\frac{5}{32}{\rm Tr}\,(\bar{L}\,\Gamma^d\Gamma^b\Gamma^c\;D_aF_{ab}\,L\,F_{cd}) - \\
&-\frac{5}{16}{\rm Tr}\,(\bar{L}\,\Gamma^c\;D_aF_{ab}\,L\,F_{cb}) - \frac{5}{16}{\rm Tr}\,(\bar{L}\,\Gamma^c\;F_{cb}\,D_aF_{ab}\,L) + \\
&+\frac{15}{16}{\rm STr}\,\Big((\bar{L}\,\Gamma^b\Gamma^c\Gamma^d\;F_{ab}\,F_{cd}\,D_aL\Big)\,.
\end{aligned}
$$

All other terms up to order $1/r^7$ come from insertions of $r\cdot K$, $\vec{K}^2$, or D_0^2. Using the symmetrization operation defined above, we can now write the complete set of two-fermion terms in the $D=0+1$ off-shell action as

$$
\begin{aligned}
\Gamma_{1/r^3} &= \Gamma^{\rm cov}_{1/r^3}\\
\Gamma_{1/r^4} &= -3\;{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; r\cdot K)\\
\Gamma_{1/r^5} &= \Gamma^{\rm cov}_{1/r^5} + \frac{1}{4}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; D_0^2) - \frac{3}{2}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; \vec{K}^2) + +\frac{15}{2}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; r\cdot K, r\cdot K)\\
\Gamma_{1/r^6} &= -5\;{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^5}; r\cdot K) - \frac{15}{8}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; D_0^2, r\cdot K) + \frac{15}{2}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; \vec{K}^2, r\cdot K)\\
\Gamma_{1/r^7} &= \Gamma^{\rm cov}_{1/r^7} + \frac{5}{8}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^5}; D_0^2) - \frac{5}{2}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^5}; \vec{K}^2) + \frac{35}{2}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^5}; r\cdot K, r\cdot K) + \\
&\quad+\frac{15}{8}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; \vec{K}^2, \vec{K}^2) - \frac{15}{16}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; D_0^2, \vec{K}^2) + \frac{1}{16}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; D_0^2, D_0^2) - \\
&\quad-\frac{105}{4}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; \vec{K}^2, r\cdot K, r\cdot K) + \frac{105}{48}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; D_0^2, r\cdot K, r\cdot K) + \\
&\quad+\frac{315}{8}{\rm Sym}\,(\Gamma^{\rm cov}_{1/r^3}; r\cdot K, r\cdot K, r\cdot K, r\cdot K)
\end{aligned}
$$

We will see that things simplify greatly when we restrict to a background that satisfies the matrix theory equations of motion. In our $D=10$ notation, these equations of motion read

$$\not{D}L = 0 \tag{A.15}$$

and

$$D_aF_{ab} = i\bar{L}\Gamma^bL\,. \tag{A.16}$$

Examining the covariant terms above, we see that apart from the final term in the $1/r^7$ action, all terms listed contain either $\not{D}L$ or D_aF_{ab}, and so these terms either cancel directly or may be canceled by the four-fermion terms which we will soon calculate.

A.3 Four-fermion terms

The remaining terms at orders up to $1/r^7$ contain four fermions. These arise from a loop with two each of the vertices (2.5) and (2.6) plus insertions of boson-boson and fermion-fermion vertices. In this part of the calculation, we will calculate using derivatives and setting K_0 to zero. Also, we will return to our 16 component spinor notation and 16×16 gamma matrices. The leading-order four-fermion terms are at $1/r^5$.

The calculations below rely heavily on Fierz rearrangement identities for the 16×16 symmetric γ matrices. A treatment of these is given in appendix B.

A.3.1 Order $1/r^5$

At order $1/r^5$ we have a single term, arising from two vertices each of the forms (2.6) and (2.5) without any derivatives. This is given by

$$\begin{aligned}\Gamma^{LLLL}_{1/r^5} &= \frac{1}{2}\int d\tau \int \frac{dk}{2\pi}\mathrm{Tr}\left(L\gamma^a \frac{\not{r}+ik}{k^2+r^2}\gamma^b L\frac{1}{k^2+r^2}L\gamma^b\frac{\not{r}+ik}{k^2+r^2}\gamma^a L\frac{1}{k^2+r^2}\right) \\ &= \frac{5}{64r^7}\int d\tau \mathrm{Tr}\,(L\gamma^a\not{r}\gamma^b L\, L\gamma^b\not{r}\gamma^a L) - \frac{1}{64r^5}\int d\tau \mathrm{Tr}\,(L\gamma^a\gamma^b L\, L\gamma^b\gamma^a L)\end{aligned} \quad (A.17)$$

Generally we would like to manipulate the gamma matrices in our expression into sums of antisymmetric products in order to apply the Fierz identities from appendix B. For the trace in the first term, we have:

$$\begin{aligned}\mathrm{Tr}\,(L\gamma^a\not{r}\gamma^b L\, L\gamma^b\not{r}\gamma^a L) &= r_i r_j\Big\{\mathrm{Tr}\,(L\gamma^{[kli]}L\, L\gamma^{[lkj]}L) + 2\mathrm{Tr}\,(L\gamma^{[ki]}L\, L\gamma^{[kj]}L) + \\ &\quad + 8\mathrm{Tr}\,(L\gamma^i L\, L\gamma^j L) + 2\mathrm{Tr}\,(L\gamma^k L\, L\gamma^k L)\delta_{ij} - \\ &\quad - 2\mathrm{Tr}\,(LL\, LL)\delta_{ij}\Big\} \\ &= r_i r_j\Big\{4\mathrm{Tr}\,(L\gamma^k L\, L\gamma^k L)\delta_{ij} - 4\mathrm{Tr}\,(LL\, LL)\delta_{ij}\Big\},\end{aligned}$$

where we have used Fierz identities (B.23,B.31) from appendix B. Note that after application of the Fierz identity, the two factors of r become contracted with one another. The second trace in (A.18) becomes

$$\begin{aligned}\mathrm{Tr}\,(L\gamma^a\gamma^b L\, L\gamma^b\gamma^a L) &= \mathrm{Tr}\,(L\gamma^{[kl]}L\, L\gamma^{[lk]}L) - 2\mathrm{Tr}\,(L\gamma^k L\, L\gamma^k L) + 10\mathrm{Tr}\,(LL\, LL) \\ &= -4\mathrm{Tr}\,(L\gamma^k L\, L\gamma^k L) + 4\mathrm{Tr}\,(LL\, LL)\end{aligned}$$

using Fierz identity (B.7). Thus, we find that the four fermion contribution at $1/r^5$ is given by

$$\begin{aligned}\Gamma^{LLLL}_{1/r^5} &= \frac{3}{8}\int d\tau(\mathrm{Tr}\,(L\gamma^i L\, L\gamma^i L) - \mathrm{Tr}\,(LL\, LL)) \\ &= -\frac{3}{8}\mathrm{Tr}\,(\bar{L}\Gamma^a L\, \bar{L}\Gamma^a L)\,.\end{aligned} \quad (A.18)$$

Now, looking back at the result for the two fermion calculation, we see that the only term up to order $1/r^5$ which does not vanish by the equations of motion is

$$\frac{3}{8}i\ \mathrm{Tr}\,(\bar{L}\Gamma^b D_a F_{ab} L)\,.$$

Bringing the last L to the front by the cyclic property of the trace and combining this with the four-fermion term we have just calculated, we have

$$-\frac{3}{8}i\ \mathrm{Tr}\,(\bar{L}\Gamma^b L\ (D_a F_{ab} - i\bar{L}\Gamma^b L))\,, \tag{A.19}$$

which clearly vanishes on-shell by the bosonic equation of motion (A.16). Thus, we have shown so far that up to order $\frac{1}{r^5}$ all terms in the one loop matrix theory effective action vanish for background fields satisfying the equations of motion.

A.3.2 Order $1/r^6$

At order $1/r^6$, we have four-fermion terms coming from taking the four mixed vertices above, plus either a single fermion vertex or a single derivative (i.e. using the first order term in the Taylor expansion of one of the background L fields). In either case, apart from r^2 factors in the denominator, each non-vanishing term contains exactly one r_i coupled to a K_i or a γ^i, so we may obtain such terms from the $1/r^7$ terms using the transformation (A.3), as described above. We will return to these terms in section (A.3.5) and show that the four-fermion terms at $1/r^6$ exactly cancel the two-fermion terms by the matrix theory equations of motion.

A.3.3 Order $1/r^7$

The contributions to the four-fermion terms at order $1/r^7$ are of four types. Firstly, we have a term with two each of the mixed vertices (2.5) and (2.6) plus a single insertion of the bosonic vertex (2.9),

$$\Gamma^{LLMLL} = \int d\tau \int \frac{dk}{2\pi}\mathrm{Tr}\,(L\gamma^a(\not{r}+ik)\gamma^b L M_{bc} L\gamma^c(\not{r}+ik)\gamma^a L)\frac{1}{(k^2+r^2)^5}\,. \tag{A.20}$$

Next, we have terms with the four mixed vertices plus two insertions of the fermion fermion vertex, either between the same pair of L's, or between different pairs of L's. The first is given by

$$\Gamma^{LKKLLL} = -\int d\tau\int \frac{dk}{2\pi}\mathrm{Tr}\,\Big(L\gamma^a(\not{r}+ik)\not{K}(\not{r}+ik)\not{K}(\not{r}+ik)\gamma^b L\ L\gamma^b(\not{r}+ik)\gamma^a L\Big)\times$$
$$\times\frac{1}{(k^2+r^2)^6}\,, \tag{A.21}$$

while the second contribution is

$$\Gamma^{LKLLKL} = -\frac{1}{2}\int d\tau\int \frac{dk}{2\pi}\mathrm{Tr}\,\Big(L\gamma^a(\not{r}+ik)\not{K}(\not{r}+ik)\gamma^b L\ L\gamma^b(\not{r}+ik)\not{K}(\not{r}+ik)\gamma^a L\Big)\times$$
$$\times\frac{1}{(k^2+r^2)^6}. \tag{A.22}$$

The remaining terms involve Taylor expansions of the background fields (see the calculation of the boson loop). For the first of these we take two each of the mixed vertices (2.5) and (2.6), as in the $1/r^5$ term, but use the second order term in the Taylor expansion of the product of background fields. This term will then have two derivatives, either on the same L or on two different Ls. Following steps analogous to the boson loop calculation above, we find that such terms are given by

$$\begin{aligned}\Gamma^{dd} = & \sum_{d_1+d_2+d_3=2} \int \frac{dk}{2\pi} \mathrm{Tr}\left(L\gamma^a \frac{\not\eta + ik}{k^2+r^2}\gamma^b L^{(d_1)} \frac{(i\partial_k)^{d_1}}{d_1!} \times \right. \\ & \left. \times \left\{ \frac{1}{k^2+r^2} L^{(d_2)}\gamma^b \frac{(i\partial_k)^{d_2}}{d_2!} \left\{ \frac{\not\eta + ik}{k^2+r^2}\gamma^a L^{(d_3)} \frac{(i\partial_k)^{d_3}}{d_3!} \left\{ \frac{1}{k^2+r^2} \right\}\right\}\right\}\right) . \end{aligned} \tag{A.23}$$

Finally, we have a term coming from taking the four mixed vertices plus a single fermionic vertex and using the first order term in the Taylor expansion (one derivative on one of the background fields). This term may be written

$$\begin{aligned}\Gamma^{Kd} = - & \sum_{d_1+d_2+d_3+d_4=1} \int \frac{dk}{2\pi} \mathrm{Tr}\left(L\gamma^a \frac{\not\eta + ik}{k^2+r^2} \not K^{(d_1)} (i\partial_k)^{d_1} \left\{ \frac{\not\eta + ik}{k^2+r^2}\gamma^b L^{(d_2)} \times \right.\right. \\ & \left.\left. \times (i\partial_k)^{d_2} \left\{ \frac{1}{k^2+r^2} L^{(d_3)}\gamma^b (i\partial_k)^{d_3} \left\{ \frac{\not\eta + ik}{k^2+r^2}\gamma^a L^{(d_4)} (i\partial_k)^{d_4} \left\{ \frac{1}{k^2+r^2} \right\}\right\}\right\}\right) . \end{aligned} \tag{A.24}$$

Before simplifying the expressions above, we note the following. Based on the calculation at $\frac{1}{r^5}$, we may expect to get four-fermion terms which pair with two-fermion terms containing $D_a F_{ab}$ to cancel them by the bosonic equations of motion (A.16). These terms are given by:

$$\begin{aligned}\Gamma^{LLLL}_{eom} = & \frac{5}{16}\mathrm{Tr}\,(\bar L \Gamma^a \vec K^2 L\ \bar L \Gamma^a L) + \frac{5}{8}\mathrm{Tr}\,(\bar L \Gamma^a L \vec K^2 \bar L \Gamma^a L) - \frac{5}{32}\mathrm{Tr}\,(\bar L \Gamma^a L\ D_0^2(\bar L)\Gamma^a L) - \\ & - \frac{5}{32}\mathrm{Tr}\,(\bar L \Gamma^a L\ \bar L \Gamma^a (D_0^2 L)) - \frac{5}{32}\mathrm{Tr}\,(\bar L \Gamma^a L\ (D_0 \bar L)\Gamma^a + (D_0 L)) + \\ & + \frac{5}{32}\mathrm{Tr}\,(\bar L \Gamma^c \Gamma^b \Gamma^d F_{cd} L\ \bar L \Gamma^b L) - \frac{5}{8}\mathrm{Tr}\,(\bar L \Gamma^a L F_{ab} \bar L \Gamma^b L) . \end{aligned} \tag{A.25}$$

We will find it convenient to start with these terms and subtract off an equivalent set of terms in the remaining calculation. We now calculate the remaining terms, proceeding by the number of derivatives appearing.

Two derivative terms We begin by calculating the terms containing two derivatives. These are exactly the terms in expression (A.24), which we may rewrite as

$$\begin{aligned} & \sum_{d_1+d_2+d_3=2} \mathrm{Tr}\,(L\gamma^a \not\eta \gamma^b L^{(d_1)}\ L^{(d_2)}\gamma^b \not\eta \gamma^a L^{(d_3)}) \times \\ & \times \int \frac{dk}{2\pi} \frac{1}{k^2+r^2} \frac{(i\partial_k)^{d_1}}{d_1!} \left\{ \frac{1}{k^2+r^2} L^{(d_2)} \frac{(i\partial_k)^{d_2}}{d_2!} \left\{ \frac{1}{k^2+r^2}\gamma^a L^{(d_3)} \frac{(i\partial_k)^{d_3}}{d_3!} \frac{1}{k^2+r^2} \right\}\right\} \end{aligned} \tag{A.26}$$

$$\begin{aligned} - & \sum_{d_1+d_2+d_3=2} \mathrm{Tr}\,(L\gamma^a \gamma^b L^{(d_1)}\ L^{(d_2)}\gamma^b \gamma^a L^{(d_3)})\} \times \\ & \times \int \frac{dk}{2\pi} \frac{k}{k^2+r^2} \frac{(i\partial_k)^{d_1}}{d_1!} \left\{ \frac{1}{k^2+r^2} L^{(d_2)} \frac{(i\partial_k)^{d_2}}{d_2!} \left\{ \frac{k}{k^2+r^2}\gamma^a L^{(d_3)} \frac{(i\partial_k)^{d_3}}{d_3!} \frac{1}{k^2+r^2} \right\}\right\} . \end{aligned} \tag{A.27}$$

Here, we have separated terms with zero ks and two ks in the numerator (the integral of terms choosing a single k in the numerator vanish). Starting with the terms (A.26), we can evaluate the integral, arrange the gamma matrices into sums of antisymmetric products and perform integration by parts to rearrange the derivatives, giving the following expression:

$$\frac{7r_ir_j}{256r^9}\Big(2\mathrm{Tr}\,(L*L\,\ddot{L}*L)+2\mathrm{Tr}\,(L*\ddot{L}\,L*L)+\mathrm{Tr}\,(L*\dot{L}\,\dot{L}*L)+\mathrm{Tr}\,(\dot{L}*\dot{L}\,L*L)\Big)\times$$
$$\times\Big\{\gamma^{[kli]}\otimes\gamma^{[lkj]}+2\gamma^{[ki]}\otimes\gamma^{[kj]}+8\gamma^i\otimes\gamma^j+2\gamma^k\otimes\gamma^k\delta_{ij}-2\mathbb{1}\otimes\mathbb{1}\delta_{ij}\Big\}\,.$$

Here, we have introduced the notation

$$\mathrm{Tr}\,(A*B\;C*D)\{\gamma\otimes\tilde{\gamma}\}=\mathrm{Tr}\,(A\gamma B\;C\tilde{\gamma}D)\,.$$

Now, using the Fierz identities (B.23), (B.31), this reduces to

$$\frac{7}{64r^7}\Big(2\mathrm{Tr}\,(L\gamma^aL\;L\gamma^a\ddot{L})+2\mathrm{Tr}\,(L\gamma^aL\;\ddot{L}\gamma^aL)+\mathrm{Tr}\,(L\gamma^a\dot{L}\;\dot{L}\gamma^aL)+\mathrm{Tr}\,(\dot{L}\gamma^a\dot{L}\;L\gamma^aL)\Big)\,. \tag{A.28}$$

We now move to the remaining terms (A.27). After integration, gamma matrix algebra, and differentiation by parts, these become:

$$\frac{-1}{256r^7}\Big(6\mathrm{Tr}\,(L*L\ddot{L}*L)+6\mathrm{Tr}\,(L*\ddot{L}L*L)+13\mathrm{Tr}\,(L*\dot{L}\dot{L}*L)+3\mathrm{Tr}\,(\dot{L}*\dot{L}L*L)\Big)\times$$
$$\times\{\gamma^{[kl]}\otimes\gamma^{[lk]}-2\gamma^k\otimes\gamma^k+10(\mathbb{1}\otimes\mathbb{1})\}\,.$$

Using the Fierz identity (B.7) we may write this as

$$\frac{3}{64r^7}\Big(2\mathrm{Tr}\,(L\gamma^aLL\gamma^a\ddot{L})+2\mathrm{Tr}\,(L\gamma^aL\ddot{L}\gamma^aL)+\mathrm{Tr}\,(L\gamma^a\dot{L}\dot{L}\gamma^aL)+\mathrm{Tr}\,(\dot{L}\gamma^a\dot{L}L\gamma^aL)\Big)-$$
$$-\frac{5}{128}\Big(\mathrm{Tr}\,(L\gamma^{[kl]}\dot{L}\;\dot{L}\gamma^{[lk]}L)-2\mathrm{Tr}\,(L\gamma^k\dot{L}\;\dot{L}\gamma^kL)+10\mathrm{Tr}\,(L\dot{L}\;\dot{L}L)\Big)\,.$$

Combining these terms with those in (A.28) and subtracting the two derivative terms in (A.25), we find the remaining two derivative term to be

$$\Gamma^{\dot{L}\dot{L}}=\frac{5}{32r^7}\Big(\mathrm{Tr}\,(L\gamma^a\dot{L}\;L\gamma^a\dot{L})+\mathrm{Tr}\,(\dot{L}\gamma^aL\;\dot{L}\gamma^aL)+2\mathrm{Tr}\,(\dot{L}\gamma^a\dot{L}\;L\gamma^aL)\Big)-$$
$$-\frac{5}{128r^7}\Big(\mathrm{Tr}\,(L\gamma^{[kl]}\dot{L}\;\dot{L}\gamma^{[lk]}L)+2\mathrm{Tr}\,(L\gamma^k\dot{L}\;\dot{L}\gamma^kL)+6\mathrm{Tr}\,(L\dot{L}\;\dot{L}L)\Big)\,. \tag{A.29}$$

Here, we have rewritten the terms with two derivatives on a single L using integration by parts to leave only terms with derivatives on separate Ls. We note here that the last line is equivalent by Fierz identity (B.7) to

$$\frac{5}{128r^7}\Big(\mathrm{Tr}\,(\dot{L}\gamma^{[kl]}\dot{L}\;L\gamma^{[lk]}L)+2\mathrm{Tr}\,(\dot{L}\gamma^k\dot{L}\;L\gamma^kL)+6\mathrm{Tr}\,(\dot{L}\dot{L}\;LL)\Big)\,, \tag{A.30}$$

however we shall use the form given above.

One derivative terms. Terms with a single derivative come either from (A.24) or from terms in (A.20) in which the indices on M are taken to be M_{0i} or M_{i0}. We begin with the latter terms. Performing the k integration in (A.20), we get

$$-\frac{35}{256}\mathrm{Tr}\left(L\gamma^a \not{r}\gamma^b LM_{bc}L\gamma^c\not{r}\gamma^a L\right)+\frac{5}{256}\mathrm{Tr}\left(L\gamma^a\gamma^b LM_b cL\gamma^c\gamma^a L\right).$$

Keeping only the terms with a derivative ($\dot{K}$), and rearranging the gamma matrices into sums of antisymmetric products, we have:

$$\frac{35}{256}\frac{r_m r_p}{r^9}\mathrm{Tr}\left(L*L\dot{K}_i L*L\right)\Big\{2\gamma^{[kmi]}\otimes\gamma^{[kp]}-2\gamma^{[km]}\otimes\gamma^{[kpi]}+4\gamma^m\otimes\gamma^{[pi]}+ \\ +4\gamma^{[mi]}\otimes\gamma^p-2\delta_{mp}(\gamma^i\otimes\mathbb{1}-\mathbb{1}\otimes\gamma^i)\Big\}+ \tag{A.31}$$

$$+\frac{5}{256r^7}\mathrm{Tr}\left(L*L\dot{K}_i L*L\right)\Big\{2\gamma^{[ki]}\otimes\gamma^k+2\gamma^k\otimes\gamma^{[ki]}-4\gamma^i\otimes\mathbb{1}+4\mathbb{1}\otimes\gamma^i\Big\}. \tag{A.32}$$

Moving to expression (A.24) we can compute the k integrals to obtain

$$\frac{5}{256}\Big(\mathrm{Tr}\left(\dot{L}\gamma^a\not{K}\gamma^b L\,L\gamma^b\gamma^a L\right)-\mathrm{Tr}\left(L\gamma^a\not{K}\gamma^b\dot{L}\,L\gamma^b\gamma^a L\right)+ \\ -3\mathrm{Tr}\left(L\gamma^a\not{K}\gamma^b L\,\dot{L}\gamma^b\gamma^a L\right)+3\mathrm{Tr}\left(L\gamma^a\not{K}\gamma^b L\,L\gamma^b\gamma^a\dot{L}\right)\Big)+ \\ +\frac{35}{256}\mathrm{Tr}\left(L\gamma^a\not{r}\dot{\not{K}}\gamma^b L\,L\gamma^b\not{r}\gamma^a L\right) \tag{A.33}$$

Here, we have also used an integration by parts to obtain the left-right antisymmetric form of the terms in the first line. From the $\dot{K}$ term in the last line, we now arrange the gamma matrices into sums of antisymmetric products, to get

$$\frac{35}{256}\frac{r_m r_p}{r^9}\mathrm{Tr}\left(L*\dot{K}_i L\,L*L\right)\Big\{-\gamma^{[klmi]}\otimes\gamma^{[lkp]}-4\gamma^{[mi]}\otimes\gamma^p+2\gamma^m\otimes\gamma^{[pi]}-2\delta_{mp}\gamma^{[ki]}\otimes\gamma^k\Big\} \tag{A.34}$$

At this point, we consider all terms containing $\dot{K}$ together, and use a Fierz identity, (B.33) to reduce these to a form in which all factors of r appear as r^2. The $\dot{K}$ terms then become

$$\frac{5}{16}\Big(\mathrm{Tr}\left(L\gamma^{[ki]}L\dot{K}_i L\gamma^k L\right)+\mathrm{Tr}\left(L\gamma^k L\dot{K}_i L\gamma^{[ki]}L\right)- \\ -2\mathrm{Tr}\left(L\gamma^i L\dot{K}_i LL\right)+2\mathrm{Tr}\left(LL\dot{K}_i L\gamma^i L\right)\Big)+ \\ +\frac{35}{128}\Big(\mathrm{Tr}\left(L\gamma^k\dot{K}_i L\,L\gamma^{[ki]}L\right)-\mathrm{Tr}\left(L\gamma^{[ki]}\dot{K}_i L\,L\gamma^k L\right)\Big)$$

Finally, subtracting those terms in (A.25) with a $\dot{K}$, we are left with

$$\Gamma^{\dot{K}}=\frac{5}{16}\Big(\mathrm{Tr}\left(L\gamma^k\dot{K}_i L\,L\gamma^{[ki]}L\right)+\mathrm{Tr}\left(L\gamma^{[ki]}L\dot{K}_i L\gamma^k L\right)+\mathrm{Tr}\left(L\gamma^k L\dot{K}_i L\gamma^{[ki]}L\right)\Big)+ \\ +\frac{5}{128}\Big(\mathrm{Tr}\left(L\gamma^{[ki]}[\dot{K}_i,L]\,L\gamma^k L\right)+\mathrm{Tr}\left(L\gamma^{[ki]}L[\dot{K}_i,L]\gamma^k L\right)\Big) \tag{A.35}$$

Going back to the $\dot{L}$ terms in (A.33), we rearrange the gamma matrices into sums of antisymmetric products and obtain

$$\Gamma^{\dot{L}} = \frac{5}{256}\Big(\mathrm{Tr}\,(L * K_i\dot{L}\; L * L) - \mathrm{Tr}\,(\dot{L} * K_i L\; L * L) + 3\mathrm{Tr}\,(L * K_i L\; \dot{L} * L) - \\ -3\mathrm{Tr}\,(L * K_i L\; L * \dot{L})\Big)\,\{\gamma^{[kli]} \otimes \gamma^{[lk]} + 6\gamma^i \otimes \mathbb{1} + 2\mathbb{1} \otimes \gamma^i\} \qquad \text{(A.36)}$$

We will use this form, however, we note that by Fierz identities (B.14, B.15), this is equivalent to

$$\frac{5}{512}\Big(\mathrm{Tr}\,(L * \dot{L}K_i L * L) - \mathrm{Tr}\,(L * LK_i\dot{L} * L) + \\ +3\mathrm{Tr}\,(\dot{L} * LK_i L * L) - 3\mathrm{Tr}\,(L * LK_i L * \dot{L})\Big) \times \\ \times \Big\{\gamma^{[kli]} \otimes \gamma^{[lk]} + \gamma^{[kl]} \otimes \gamma^{[lki]} + 4\gamma^{[ki]} \otimes \gamma^k - 4\gamma^k \otimes \gamma^{[ki]} + 8\gamma^i \otimes \mathbb{1} + 8\mathbb{1} \otimes \gamma^i\Big\}$$

Zero derivative terms Finally, we calculate the terms with no derivatives. These come from the term with a boson-boson vertex, which we found above to be

$$-\frac{35}{256}\mathrm{Tr}\,(L\gamma^a \not{\eta}\gamma^b L M_{bc} L\gamma^c \not{\eta}\gamma^a L) + \frac{5}{256}\mathrm{Tr}\,(L\gamma^a\gamma^b L M_{bc} L\gamma^c\gamma^a L) \qquad \text{(A.37)}$$

plus the two terms with a pair of fermion-fermion vertices. Performing the k integrals in these, (A.21) becomes

$$-\frac{35}{256}\mathrm{Tr}\,(L\gamma^a \not{\eta}\not{K}\not{K}\gamma^b L\; L\gamma^b \not{\eta}\gamma^a L) + \frac{5}{256}\mathrm{Tr}\,(L\gamma^a \not{K}\not{K}\gamma^b L\; L\gamma^b\gamma^a L)\,, \qquad \text{(A.38)}$$

while (A.22) gives simply

$$\frac{5}{64}\mathrm{Tr}\,(L\gamma^a \not{K}\gamma^b L\; L\gamma^b \not{K}\gamma^a L)\,. \qquad \text{(A.39)}$$

Rewriting this last term in our standard form with antisymmetric products of gamma matrices, we get

$$\frac{5}{64}\mathrm{Tr}\,(L * K_i L\; L * K_j L)\Big\{\gamma^{[kli]} \otimes \gamma^{[lkj]} + 2\gamma^{[ki]} \otimes \gamma^{[kj]} + \\ + 6\gamma^i \otimes \gamma^j + 2\gamma^j \otimes \gamma^i + 2\gamma^k \otimes \gamma^k \delta_{ij} - 2\mathbb{1} \otimes \mathbb{1}\delta_{ij}\Big\} \qquad \text{(A.40)}$$

We will return to this expression later.

Since terms in expressions (A.37) and (A.38) may be related by Fierz identities, we will deal with them together. We first note that since

$$M_{ab} = \vec{K}^2\delta_{ab} + 2iF_{ab}$$

and

$$\not{K}\not{K} = \vec{K}^2 - \frac{i}{2}F_{ij}\gamma^i\gamma^j\,,$$

that terms in (A.37) and (A.38) split naturally into terms involving F and terms involving $\vec{K}^2$. After some gamma matrix algebra, the $\vec{K}^2$ terms become

$$\frac{5}{256r^7}\Big(\mathrm{Tr}\,(L*L\vec{K}^2L*L)+\mathrm{Tr}\,(L*\vec{K}^2L\;L*L)\Big)\times$$
$$\times\Big\{\gamma^{[kl]}\otimes\gamma^{[lk]}-2\gamma^k\otimes\gamma^k+10\mathbb{1}\otimes\mathbb{1}\Big\}-$$
$$-\frac{35}{256}\frac{r_ir_j}{r^9}\Big(\mathrm{Tr}\,(L*L\vec{K}^2L*L)+\mathrm{Tr}\,(L*\vec{K}^2L\;L*L)\Big)\times$$
$$\times\Big\{\gamma^{[kli]}\otimes\gamma^{[lkj]}+2\gamma^{[ki]}\otimes\gamma^{[kj]}+8\gamma^i\otimes\gamma^j+2\delta_{ij}(\gamma^k\otimes\gamma^k-\mathbb{1}\otimes\mathbb{1})\Big\}$$

We may now apply Fierz identities (B.23), (B.31) and (B.7) to reduce all of these terms to simply

$$-\frac{5}{8}\left(\mathrm{Tr}\,(L\gamma^a\vec{K}^2LL\gamma^aL+\mathrm{Tr}\,(L\gamma^aL\vec{K}^2L\gamma^aL)\right).$$

Subtracting off the terms containing $\vec{K}^2$ in (A.25), we are left with only

$$\Gamma^{K^2}=-\frac{5}{16}\mathrm{Tr}\,(L\gamma^aL\;L\gamma^a\vec{K}^2L)\,. \tag{A.41}$$

At this point, we note that (A.41) and (A.40) may be combined to give a term

$$\Gamma^{\mathrm{cov}}=\frac{5}{32}\Big\{\mathrm{Tr}\,(L\gamma^a[K_i,L]\;L\gamma^a[K_i,L])+\mathrm{Tr}\,(K_i,L]\gamma^aL\;[K_i,L]\gamma^aL)+$$
$$+2\mathrm{Tr}\,(L\gamma^aL\;[K_i,L]\gamma^a[K_i,L])\Big\} \tag{A.42}$$

which combines naturally with (A.29) to yield a $D=10$ covariant term, plus a remaining term

$$\Gamma^{LKL}=\frac{5}{64}\mathrm{Tr}\,(L*K_iL\;L*K_jL)\Big\{\gamma^{[kli]}\otimes\gamma^{[lkj]}+2\gamma^{[ki]}\otimes\gamma^{[kj]}+6\gamma^i\otimes\gamma^j+$$
$$+2\gamma^j\otimes\gamma^i-2\gamma^k\otimes\gamma^k\delta_{ij}+2\mathbb{1}\otimes\mathbb{1}\delta_{ij}\Big\}.$$

By the Fierz identities (B.23, B.29), we see that this has an equivalent form,

$$\frac{5}{64}\mathrm{Tr}\,(L*LK_iL*LK_j)\Bigg\{-\frac{1}{2}\gamma^{[kli]}\otimes\gamma^{[lkj]}-\frac{1}{2}\gamma^{[klj]}\otimes\gamma^{[lki]}-\gamma^{[ki]}\otimes\gamma^{[kj]}-$$
$$-\gamma^{[kj]}\otimes\gamma^{[ki]}-4\gamma^i\otimes\gamma^j-4\gamma^j\otimes\gamma^i-4\gamma^{[kij]}\otimes\gamma^k+$$
$$+4\gamma^{[ij]}\otimes\mathbb{1}+2\gamma^k\otimes\gamma^k\delta_{ij}-2\mathbb{1}\otimes\mathbb{1}\delta_{ij}\Bigg\}$$

We now return to the terms in (A.37) and (A.38) invloving F_{ij}. From (A.37), we have F terms given by

$$-\frac{35ir_mr_p}{256r^9}\mathrm{Tr}\,(L*LF_{ij}L*L)\times$$
$$\times\Big\{2\gamma^{[kmi]}\otimes\gamma^{[kpj]}+2\gamma^{[mij]}\otimes\gamma^p+2\gamma^m\otimes\gamma^{[pij]}-2\gamma^{[mi]}\otimes\gamma^{[pj]}-2\gamma^i\otimes\gamma^j\delta_{mp}\Big\}-$$
$$-\frac{5i}{256r^7}\mathrm{Tr}\,(L*LF_{ij}L*L)\Big\{-2\gamma^{[ki]}\otimes\gamma^{[kj]}-2\gamma^{[ij]}\otimes\mathbb{1}-2\mathbb{1}\otimes\gamma^{[ij]}-2\gamma^i\otimes\gamma^j\Big\},$$

while from (A.38), the corresponding terms are

$$-\frac{35ir_mr_p}{256r^9}\mathrm{Tr}\,(L*F_{ij}L\;L*L)\Big\{-\frac{1}{2}\gamma^{[klmij]}\otimes\gamma^{[lkp]}-\gamma^{[kmij]}\otimes\gamma^{[kp]}-2\gamma^{[mij]}\otimes\gamma^{p}+$$
$$+\gamma^{m}\otimes\gamma^{[pij]}-\delta_{mp}(\gamma^{[kij]}\otimes\gamma^{k}-\gamma^{[ij]}\otimes\mathbb{1})\Big\}-$$
$$-\frac{5i}{256r^7}\mathrm{Tr}\,(L*F_{ij}L\;L*L)\Big\{\frac{1}{2}\gamma^{[klij]}\otimes\gamma^{[lk]}-\gamma^{[kij]}\otimes\gamma^{k}-\mathbb{1}\otimes\gamma^{[ij]}+3\gamma^{[ij]}\otimes\mathbb{1}\Big\}$$

Considering together all terms from these expressions in the 35/256 brackets, we may use Fierz identity (B.33) to reduce these to a form with all factors of r_i appearing as r^2. Simplifying the remaining terms in the 5/256 brackets using Fierz identities (B.26) and (B.28), we find that these reduce to a set of terms proportional to the 35/256 terms, and combining all F terms, we get

$$-\frac{5i}{32}\mathrm{Tr}\,(L*F_{ij}L\;L*L)\Big\{\gamma^{k}\otimes\gamma^{[kij]}-\gamma^{[kij]}\otimes\gamma^{k}+\mathbb{1}\otimes\gamma^{[ij]}-\gamma^{[ij]}\otimes\mathbb{1}\Big\}-$$
$$-\frac{5i}{32}\mathrm{Tr}\,(L*LF_{ij}L*L)\Big\{\gamma^{k}\otimes\gamma^{[kij]}+\gamma^{[kij]}\otimes\gamma^{k}-\mathbb{1}\otimes\gamma^{[ij]}-\gamma^{[ij]}\otimes\mathbb{1}-4\gamma^{i}\otimes\gamma^{j}\Big\}.$$

Finally, we note that the only terms from (A.25) that we have not yet accounted for contain F_{ij}, and subtracting these off now, we are left with

$$\begin{aligned}\Gamma^{F}=-\frac{5i}{32}\Big\{&\mathrm{Tr}\,(L\gamma^{k}F_{ij}L\;L\gamma^{[kij]}L)+\mathrm{Tr}\,(L\gamma^{[kij]}LF_{ij}L\gamma^{k}L)+\mathrm{Tr}\,(L\gamma^{k}LF_{ij}L\gamma^{[kij]}L)+\\&-\mathrm{Tr}\,(LF_{ij}L\;L\gamma^{[ij]}L)-\mathrm{Tr}\,(L\gamma^{[ij]}LF_{ij}LL)+\mathrm{Tr}\,(LLF_{ij}L\gamma^{[ij]}L)\Big\}.\end{aligned}\tag{A.43}$$

These terms combine with the $\dot{K}$ terms (A.35) into a $D=10$ covariant expression which we write below.

A.3.4 Summary of four-fermion terms

We now review the four-fermion terms calculated above. Firstly, we have a set of terms (A.25) which can be obtained by replacing D_aF_{ab} with $D_aF_{ab}-\bar{L}\Gamma^aL$ everywhere it appears in the two fermion action, and thus cancel the corresponding two-fermion terms when the background fields satisfy the equations of motion. Collecting all the remaining terms, we define

$$\begin{aligned}\Gamma^{4L}_{1/r^7}&=\Gamma^{\dot{L}\dot{L}}+\Gamma^{\dot{L}}+\Gamma^{\dot{K}}+\Gamma^{\mathrm{cov}}+\Gamma^{F}+\Gamma^{LKL}\\&=\frac{5}{32}\Big\{\mathrm{Tr}\,(\bar{L}\Gamma^aD_bL\;\bar{L}\Gamma^aD_bL)+\mathrm{Tr}\,(D_b\bar{L}\Gamma^aL\;D_b\bar{L}\Gamma^aL)+\\&\qquad+2\mathrm{Tr}\,(\bar{L}\Gamma^aL\;D_b\bar{L}\Gamma^aD_bL)\Big\}+\end{aligned}\tag{A.44}$$

$$+\frac{5i}{32}\Big\{\mathrm{Tr}\,(\bar{L}\Gamma^c F_{ab}L\;\bar{L}\Gamma^{[cab]}L) + \mathrm{Tr}\,(\bar{L}\Gamma^{[cab]}LF_{ab}\bar{L}\Gamma^c L) + \\ + \mathrm{Tr}\,(\bar{L}\Gamma^c\bar{L}F_{ab}L\Gamma^{[cab]}L)\Big\} +$$

$$+\frac{5}{128}\Big\{\{\mathrm{Tr}\,(L\gamma^{[ki]}[\dot{K}_i, L]\;L\gamma^k L) + \mathrm{Tr}\,(L\gamma^{[ki]}, L\;[\dot{K}_i L]\gamma^k L)\Big\} - \tag{A.45}$$

$$-\frac{5}{128}\Big\{\mathrm{Tr}\,(L\gamma^{[kl]}\dot{L}\;\dot{L}\gamma^{[lk]}L) + 2\mathrm{Tr}\,(L\gamma^k\dot{L}\;\dot{L}\gamma^k L) + 6\mathrm{Tr}\,(L\dot{L}\;\dot{L}L)\Big\} + \tag{A.46}$$

$$+\frac{5}{256}\Big\{\mathrm{Tr}\,(L\gamma^{[kli]}K_i\dot{L}\;L\gamma^{[lk]}L) + 6\mathrm{Tr}\,(L\not{K}\dot{L}\;LL) + 2\mathrm{Tr}\,(LK_i\dot{L}\;L\gamma^i L) - \\ - \mathrm{Tr}\,(\dot{L}\gamma^{[kli]}K_iL\;L\gamma^{[lk]}L) - 6\mathrm{Tr}\,(\dot{L}\not{K}L\;LL) - 2\mathrm{Tr}\,(\dot{L}K_iL\;L\gamma^i L)\Big\} + \\ +\frac{15}{256}\Big\{\mathrm{Tr}\,(L\gamma^{[kli]}K_iL\;\dot{L}\gamma^{[lk]}L) + 6\mathrm{Tr}\,(L\not{K}L\;\dot{L}L) + \\ + 2\mathrm{Tr}\,(LK_iL\;\dot{L}\gamma^i L) - \mathrm{Tr}\,(L\gamma^{[kli]}K_iL\;L\gamma^{[lk]}\dot{L}) - \\ - 6\mathrm{Tr}\,(L\not{K}L\;L\dot{L}) - 2\mathrm{Tr}\,(LK_iL\;L\gamma^i\dot{L})\Big\} + \tag{A.47}$$

$$+\frac{5}{64}\Big\{\mathrm{Tr}\,(L\gamma^{[kli]}K_iL\;L\gamma^{[lkj]}K_jL) + 2\mathrm{Tr}\,(L\gamma^{[ki]}K_iL\;L\gamma^{[kj]}K_jL) + \\ + 6\mathrm{Tr}\,(L\gamma^iK_iL\;L\gamma^jK_jL) + 2\mathrm{Tr}\,(L\gamma^jK_iL\;L\gamma^iK_jL) - \\ - 2\mathrm{Tr}\,(L\gamma^iK_jL\;L\gamma^iK_jL) + 2\mathrm{Tr}\,(LK_iL\;LK_iL)\Big\} \tag{A.48}$$

One might hope that these terms might all be combined into a $D = 10$ covariant expression, however, this does not appear to be the case. In particular, one can show that there is no way to rewrite the above expression in a form with all K's appearing in commutators.

A.3.5 Order $1/r^6$ terms

To complete the demonstration of the cancellation of the on-shell action below order $1/r^7$ we now calculate the terms at order $1/r^6$ using the transformation (A.3) on the $1/r^7$ terms, as described above. From the terms (A.25) we find $r \cdot K$ terms given by

$$\frac{5}{8}\mathrm{Tr}\,(\bar{L}\Gamma^a r\cdot KL\;\bar{L}\Gamma^a L) + \frac{5}{4}\mathrm{Tr}\,(\bar{L}\Gamma^a Lr\cdot K\bar{L}\Gamma^a L)\,. \tag{A.49}$$

These are exactly the terms obtained by replacing D_aF_{ab} with $D_aF_{ab} - i\bar{L}\Gamma^b L$ in the two fermion $1/r^6$ terms, providing cancellation on-shell. We have additional terms at $1/r^6$ from applying our transformation to (A.47) and (A.48), however by Fierz identities (B.14), (B.15), (B.23), and (B.29), these may be written in a form with $\not{D}L$ in each term, and therefore vanish on-shell.

This completes our calculation.

B. Fierz identities

In manipulating four-fermion terms in our answer, it is necessary to make use of various Fierz rearrangement identities which shall now be derived. All identities are based on the fact that $1, \gamma^i, \gamma^{[ij]}, \gamma^{[ijk]}$, and $\gamma^{[ijkl]}$ $i, j, k, l \in \{1, \ldots, 9\}$ form a complete basis of the 16×16 matrices. This may be expressed by the completeness relation:

$$\delta_{\alpha\beta}\delta_{\gamma\delta} = \sum_{n=0}^{4} \frac{1}{16n!} \gamma^{[a_1 \ldots a_n]}_{\beta\delta} \gamma^{[a_n \ldots a_1]}_{\gamma\alpha} \tag{B.1}$$

or equivalently:

$$M = \sum_{n=0}^{4} \frac{1}{16n!} \mathrm{Tr}\,(M\gamma^{[a_1 \ldots a_n]})\gamma^{[a_n \ldots a_1]}\,,$$

for an arbitrary matrix M. Note in particular that

$$\gamma^{[a_1 \ldots a_{9-n}]} = \frac{1}{n!}\epsilon^{(a_1 \ldots a_{9-n} b_1 \ldots b_n)}\gamma^{[b_n \ldots b_1]}\,. \tag{B.2}$$

We will derive a large class of Fierz identities as follows. Consider a term

$$f_n \equiv \mathrm{Tr}\,\left(A\gamma_1\gamma^{[a_1 \ldots a_n]}\gamma_2 B\; C\gamma_3\gamma^{[a_n \ldots a_1]}\gamma_4 D\right),$$

where A, B, C, and D are $N \times N$ matrices with a single fermionic index and γ_i are arbitrary products of gamma matrices whose spatial indices are not contracted with other gamma matrices. Then we may write a general non-abelian four-fermion term as a sum of such terms, since we may always rearrange the individual gamma matrices to get a sum of terms in which gamma matrices with contracted indices appear together in antisymmetrized products as in f_n. By relation (B.2), we need only consider $n \leq 4$. The terms f_n are certainly not all independent (even without appealing to Fierz identities), however, choosing this form will be useful for our derivation. Now we will apply our general Fierz relation (B.1). We have:

$$\begin{aligned} f_n &= \mathrm{Tr}\,\left((A\gamma_1)_\alpha \delta_{\alpha\beta}(\gamma^{[a_1 \ldots a_n]}\gamma_2 B)_\beta\; (C\gamma_3\gamma^{[a_n \ldots a_1]})_\gamma \delta_{\gamma\delta}(\gamma_4 D)_\delta\right) \\ &= -\sum_{m=0}^{4} \frac{1}{16m!} \mathrm{Tr}\,\left(B\gamma_2^T\gamma^{[a_n \ldots a_1]}\gamma^{[b_1 \ldots b_m]}\gamma^{[a_1 \ldots a_n]}\gamma_3^T C\; D\gamma_4^T\gamma^{[b_m \ldots b_1]}\gamma_1^T A\right), \end{aligned}$$

where the minus sign is due to an odd rearrangement of fermionic matrices in the trace. To simplify this, we may use the following identities, which can be easily checked:

$$\gamma^{[a_n \ldots a_1]}\gamma^{[b_1 \ldots b_m]}\gamma^{[a_1 \ldots a_n]} = c_{n,m}\gamma^{[b_1 \ldots b_m]}\,,$$

where

$$
\begin{aligned}
c_{0,m} &= 1\,, \qquad c_{1,m} = (-1)^m(9-2m) \qquad c_{2,m} = \{(9-2m)^2 - 9\} \\
c_{3,m} &= \{(-1)^m(9-2m)^3 - 25(-1)^m(9-2m)\} \\
c_{4,m} &= \{(9-2m)^4 - 46(9-2m)^2 + 189\}\,.
\end{aligned}
$$

Then we have:

$$
f_n = -\sum_{m=0}^{4} \frac{c_{n,m}}{16 m!} g_m
$$

or simply

$$
\vec{f} = -\mathbf{M}\vec{g}\,, \tag{B.3}
$$

where we have defined

$$
g_n \equiv \mathrm{Tr}\left(B\gamma_2^T\gamma^{[a_1\ldots a_n]}\gamma_3^T C\ D\gamma_4^T\gamma^{[a_n\ldots a_1]}\gamma_1^T A\right)
$$

and

$$
\mathbf{M}_{nm} = \frac{c_{n,m}}{16 m!} = \begin{bmatrix}
\frac{1}{16} & \frac{1}{16} & \frac{1}{32} & \frac{1}{96} & \frac{1}{384} \\
\frac{9}{16} & \frac{-7}{16} & \frac{5}{32} & \frac{-1}{32} & \frac{1}{384} \\
\frac{9}{2} & \frac{5}{2} & \frac{1}{2} & 0 & \frac{-1}{48} \\
\frac{63}{2} & \frac{-21}{2} & 0 & \frac{1}{2} & \frac{-1}{16} \\
189 & 21 & \frac{-21}{2} & \frac{-3}{2} & \frac{3}{8}
\end{bmatrix}.
$$

Note that g_n and f_n have the same form, and more importantly, that we may follow the same steps to show that

$$
\vec{g} = -\mathbf{M}\vec{f}\,. \tag{B.4}
$$

Now, (B.3) and (B.4) are equivalent to the relations

$$
\begin{aligned}
(\vec{f} + \vec{g}) &= -\mathbf{M}(\vec{f} + \vec{g}) \\
(\vec{f} - \vec{g}) &= \mathbf{M}(\vec{f} - \vec{g})
\end{aligned}
$$

or

$$
\begin{aligned}
(\mathbb{1} + \mathbf{M})(\vec{f} + \vec{g}) &= 0 \\
(\mathbb{1} - \mathbf{M})(\vec{f} - \vec{g}) &= 0\,.
\end{aligned}
$$

The matrix $(\mathbb{1} + \mathbf{M})$ has rank 3, and the space of its rows is spanned by the vectors:

$$
\begin{aligned}
\vec{v}_4 &= (1, 0, 0, 24, 216) \\
\vec{v}_3 &= (0, 1, 0, -6, 30) \\
\vec{v}_2 &= (0, 0, 1, 2, 6)\,.
\end{aligned}
$$

The matrix $(\mathbb{1} - \mathbf{M})$ has rank 2, and the space of its rows is spanned by the vectors:

$$
\begin{aligned}
\vec{u}_4 &= (1, 0, 24, -120, -216) \\
\vec{u}_3 &= (0, 1, -3, 36, -36)\,.
\end{aligned}
$$

Thus, we have five basic relations, given by

$$\vec{v}_i \cdot (\vec{f} + \vec{g}) \,, \tag{B.5}$$

for $(i = 2, 3, 4)$ and

$$\vec{u}_j \cdot (\vec{f} + \vec{g}) \,, \tag{B.6}$$

for $(j = 3, 4)$ which should generate all of our Fierz identities.

The identities we have derived provide relations among terms of the form f_n, but we may write the set of terms of this form in terms of a smaller set of terms of a standard form, for example, terms of the form:

$$\mathrm{Tr}\left(A\gamma^{[a_1\ldots a_n b_1\ldots b_k]}B\; C\gamma^{[a_n\ldots a_1 c_1\ldots c_m]}D\right).$$

Using the the relation (B.2) we may further restrict our basis to terms with $n+k \leq 4$ and $n + m \leq 4$. With these transformations, each of the relations above reduces to a Fierz identity relating elements of this smaller basis.

We will mainly be interested in relations involving terms with a small number of free indices. Here, we will derive the complete set of relations involving zero, one or two free indices, plus particular relations for three and four free indices that are needed in our calculation.

B.1 Zero free indices

For the case of zero free indices, we have $\gamma_i = 1$, and all of the terms f_n are already in a standard form. Thus, in this case, we simply have the five relations (B.5, B.6) above. Explicitly, defining

$$\begin{aligned} C_n &= \mathrm{Tr}\left(A\gamma^{[a_1\ldots a_n]}BC\gamma^{[a_n\ldots a_1]}D\right) \\ D_n &= \mathrm{Tr}\left(B\gamma^{[a_1\ldots a_n]}CD\gamma^{[a_n\ldots a_1]}A\right) \end{aligned}$$

we have

$$(C_2 + D_2) + 2(C_1 + D_1) + 6(C_0 + D_0) = 0 \tag{B.7}$$

$$(C_3 + D_3) - 6(C_1 + D_1) + 30(C_0 + D_0) = 0 \tag{B.8}$$

$$(C_4 + D_4) + 24(C_1 + D_1) + 216(C_0 + D_0) = 0 \tag{B.9}$$

$$(C_3 - D_3) - 3(C_2 - D_2) + 36(C_1 + D_1) - 36(C_0 + D_0) = 0 \tag{B.10}$$

$$(C_4 - D_4) + 24(C_2 - D_2) - 120(C_1 + D_1) - 216(C_0 + D_0) = 0\,. \tag{B.11}$$

As an example, for the case $A = B = C = D = L$, we see that $C_n = D_n$, so the last two relations are trivial, and and we have three independent Fierz identities. Explicitly, the relation (B.7) is

$$\mathrm{Tr}\,(L\gamma^{[ij]}L\; L\gamma^{[ji]}L) + 2\mathrm{Tr}\,(L\gamma^i L\; L\gamma^i L) + 6\mathrm{Tr}\,(LL\; LL) = 0\,.$$

Note that except in this case of equal A, B, C, and D, the Fierz identities will always relate terms with one ordering of the matrices to terms with another ordering.

B.2 One free index

For the case of one free index, there are four possible choices for f_n, corresponding to taking three of the matrices $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ equal to the identity and the fourth equal to γ^i. For each of these choices, we have five relations (B.5), (B.6), however, not all twenty of these relations are independent. In order to rewrite the Fierz identities in terms of our standard basis with only antisymmetrized products of four or less gamma matrices, we can use the relations

$$\gamma^i\gamma^{[a_1\ldots a_n]} \otimes \gamma^{[a_n\ldots a_1]} = (-1)^n\gamma^{[a_1\ldots a_n i]} \otimes \gamma^{[a_n\ldots a_1]} + n\gamma^{[a_2\ldots a_n]} \otimes \gamma^{[a_n\ldots a_2 i]} \qquad \text{(B.12)}$$

and its mirror image

$$\gamma^{[a_1\ldots a_n]}\gamma^i \otimes \gamma^{[a_n\ldots a_1]} = \gamma^{[a_1\ldots a_n i]} \otimes \gamma^{[a_n\ldots a_1]} + (-1)^{n-1}n\gamma^{[a_2\ldots a_n]} \otimes \gamma^{[a_n\ldots a_2 i]}\,.$$

By (B.2), we note also that

$$\gamma^{[klmni]} \otimes \gamma^{[nmlk]} = \gamma^{[klmn]} \otimes \gamma^{[nmlki]} = \frac{1}{4!}\epsilon^{(wxyzklmni)}\gamma^{[zyxw]} \otimes \gamma^{[klmn]} \qquad \text{(B.13)}$$

so we see that certain relations in the set we are considering (those corresponding to v_4 and u_4) will actually relate terms in our standard basis containing one free index to terms containing eight free indices, via the nine-index epsilon tensor.

We now define

$$\begin{aligned}
E_n &= \mathrm{Tr}\,(A\gamma^{[a_1\ldots a_n i]}BC\gamma^{[a_n\ldots a_1]}D)\\
\bar{E}_n &= \mathrm{Tr}\,(A\gamma^{[a_1\ldots a_n]}BC\gamma^{[a_n\ldots a_1 i]}D)\\
F_n &= \mathrm{Tr}\,(B\gamma^{[a_1\ldots a_n i]}CD\gamma^{[a_n\ldots a_1]}A)\\
\bar{F}_n &= \mathrm{Tr}\,(B\gamma^{[a_1\ldots a_n]}CD\gamma^{[a_n\ldots a_1 i]}A)\,,
\end{aligned}$$

where the bar denotes swapping the two sets of gamma matrices in a term. Then writing the basic relations (B.5) for each of our four choices of f_n give:

$$\begin{aligned}
v_i^n\Big((-1)^n E_n^+ + nE_{n-1}^+ + F_n^+ + (-1)^{n-1}nF_{n-1}^+\Big) &= 0\\
v_i^n\Big((-1)^n E_n^- - nE_{n-1}^- - F_n^- + (-1)^{n-1}nF_{n-1}^-\Big) &= 0\\
v_i^n\Big(E_n^+ + (-1)^{n-1}nE_{n-1}^+ + (-1)^n F_n^+ + nF_{n-1}^+\Big) &= 0\\
v_i^n\Big(E_n^- + (-1)^n nE_{n-1}^- + (-1)^n F_n^- - nF_{n-1}^-\Big) &= 0\,.
\end{aligned}$$

Here we have written the relations in linear combinations involving only $E_n^+ \equiv E_n + \bar{E}_n$ and $E_n^- \equiv E_n - \bar{E}_n$. From (B.6), we get a set of relations of identical form with v replaced by u and the signs on all B terms reversed.

It remains only to write these twenty relations explicitly for each v and u, and determine a linearly independent set. This is easily done, and we find the following nine independent Fierz identities:

$$(E_2^+ + F_2^+) + 8(E_0^+ + F_0^+) = 0 \tag{B.14}$$
$$E_2^- + 4E_0^- - 4F_1^- = 0 \tag{B.15}$$
$$F_2^- + 4F_0^- + 4E_1^- = 0 \tag{B.16}$$
$$(E_3^+ + F_3^+) + 42(E_1^+ + F_1^+) = 0 \tag{B.17}$$
$$(E_3^+ - F_3^+) - 6(E_1^+ - F_1^+) = 0 \tag{B.18}$$
$$E_3^- + 6E_1^- + 48F_0^- = 0 \tag{B.19}$$
$$F_3^- + 6F_1^- + 48E_0^- = 0 \tag{B.20}$$
$$P + Q + 120(E_0^+ + F_0^+) = 0 \tag{B.21}$$
$$P - Q + 12(E_2^+ - F_2^+) - 168(E_0^+ - F_0^+) = 0\,. \tag{B.22}$$

In the last two identities, we have defined

$$P \equiv E_4 = \bar{E}_4 = \frac{1}{4!}\epsilon^{(wxyzklmni)}\mathrm{Tr}\,(A\gamma^{[wxyz]}BC\gamma^{[klmn]}D)$$

and

$$Q \equiv F_4 = \bar{F}_4 = \frac{1}{4!}\epsilon^{(wxyzklmni)}\mathrm{Tr}\,(B\gamma^{[wxyz]}CD\gamma^{[klmn]}A)\,.$$

B.3 Two free indices

Here we proceed exactly as in the preceding section. In this case, there are sixteen independent ways to choose f_n to have two free indices on gamma matrices. For each of these, we get Fierz identities from (B.5) and (B.6). We can use identities analogous to (B.12) to rewrite these in terms of our standard basis. The relevant basis elements in this case are

$$\begin{aligned}
A_n &= \mathrm{Tr}\,(A\gamma^{[a_1\ldots a_n ij]}BC\gamma^{[a_n\ldots a_1]}D) \\
a_n &= \mathrm{Tr}\,(A\gamma^{[a_1\ldots a_n i]}BC\gamma^{[a_n\ldots a_1 j]}D) \\
B_n &= \mathrm{Tr}\,(B\gamma^{[a_1\ldots a_n ij]}CD\gamma^{[a_n\ldots a_1]}A) \\
B_n &= \mathrm{Tr}\,(B\gamma^{[a_1\ldots a_n i]}CD\gamma^{[a_n\ldots a_1 j]}A) \\
c_n &= C_n\delta_{ij} \\
d_n &= D_n\delta_{ij}
\end{aligned}$$

plus the corresponding quantities $\bar{A}_n, \bar{a_n}, \bar{B}_n$ and $\bar{b_n}$ with gamma matrices swapped. Using these definitions, the Fierz identities relating quantities with two free indices are then given by

$$v_n\Big((-1)^n a_n^+ + (-1)^{n-1}n(n-1)a_{n-2}^+ + 2nb_{n-1}^+ + 2nc_{n-1} + 2(-1)^n d_n\Big) = 0$$

plus all other relations derived from this by one or more of the transformations

$$(A \leftrightarrow B,\ a \leftrightarrow b,\ c \leftrightarrow d)$$
$$(a \leftrightarrow A,\ b \leftrightarrow -B,\ c \to 0,\ d \to 0)$$
$$(a_n^+ \to (-1)^n a_n^-,\ b_n^+ \to (-1)^n B_n^-,\ c_n \to 0,\ d_n \to 0)$$
$$(v \to u,\ B \to -B,\ b \to -b,\ d \to -d)\,.$$

Here, we write explicitly only the relations corresponding to v_2, which we find to be

$$0 = a_2^+ - 2a_1^+ + 4a_0^+ + 4b_1^+ + 4b_0^+ + 4c_1 + 4c_0 + 2d_2 - 4d_1 + 12d_0 \tag{B.23}$$
$$0 = b_2^+ - 2b_1^+ + 4b_0^+ + 4a_1^+ + 4a_0^+ + 4d_1 + 4d_0 + 2c_2 - 4c_1 + 12c_0 \tag{B.24}$$
$$0 = A_2^+ - 2A_1^+ + 4A_0^+ - 4B_1^+ - 4B_0^+ \tag{B.25}$$
$$0 = B_2^+ - 2B_1^+ + 4B_0^+ - 4A_1^+ - 4A_0^+ \tag{B.26}$$
$$0 = A_2^- + 2A_1^- + 4A_0^- + 4b_1^- - 4b_0^- \tag{B.27}$$
$$0 = B_2^- + 2B_1^- + 4B_0^- - 4a_1^- + 4a_0^- \tag{B.28}$$
$$0 = a_2^- + 2a_1^- + 4a_0^- - 4B_1^- + 4B_0^- \tag{B.29}$$
$$0 = b_2^- + 2b_1^- + 4b_0^- + 4A_1^- - 4A_0^- \tag{B.30}$$
$$0 = (c_2 + d_2) + 2(c_1 + d_1) + 6(c_0 + d_0)\,. \tag{B.31}$$

The last of these is simply relation (B.7) multiplied by δ_{ij}.

B.4 Three free indices

We will need only a single Fierz identity with three free indices, in order to eliminate terms at $1/r^7$ with two $\not\eta$s and a $\dot{K}_i$. By a judicious choice of f_n, we can generate the required identity. Taking

$$\begin{aligned} f_n = {} & r_m r_p \mathrm{Tr}\left(L\gamma^m\gamma^{[a_1\ldots a_n]}\gamma^i \dot{K}_i L\; L\gamma^{[a_n\ldots a_1]}\gamma^p L\right) + \\ & + r_m r_p \mathrm{Tr}\left(L\gamma^i\gamma^{[a_1\ldots a_n]}\gamma^m \dot{K}_i L\; L\gamma^p\gamma^{[a_n\ldots a_1]} L\right), \end{aligned}$$

we have

$$\begin{aligned} g_n = {} & r_m r_p \mathrm{Tr}\left(L\gamma^n\gamma^{[a_1\ldots a_n]}\gamma^m L\dot{K}_i L\gamma^i\gamma^{[a_n\ldots a_1]} L\right) + \\ & + r_m r_p \mathrm{Tr}\left(L\gamma^{[a_1\ldots a_n]}\gamma^i L\dot{K}_i L\gamma^m\gamma^{[a_n\ldots a_1]}\gamma^n L\right), \end{aligned}$$

and the required relation is $v_2^n(f_n + g_n) = 0$. This becomes (scaling by $1/2$ and ignoring $r \cdot K$ terms),

$$\begin{aligned} 0 = {} & r_m r_p \mathrm{Tr}\,(L * \dot{K}_i L\; L * L)\Big\{\gamma^{[klmi]} \otimes \gamma^{[lkp]} + 4\gamma^{[mi]} \otimes \gamma^p - 2\gamma^m \otimes \gamma^{[pi]} + \\ & \qquad + 2\delta_{mp}\gamma^{[ki]} \otimes \gamma^k - 2\delta_{mp}(\gamma^{[ki]} \otimes \gamma^k - 2\gamma^k \otimes \gamma^{[ki]})\Big\} + \\ & + r_m r_p \mathrm{Tr}\,(L * L\dot{K}_i L * L)\Big\{-2\gamma^{[kmi]} \otimes \gamma^{[kp]} + 2\gamma^{[km]} \otimes \gamma^{[kpi]} - \\ & \qquad - 4\gamma^{[mi]} \otimes \gamma^p - 4\gamma^m \otimes \gamma^{[pi]} + \\ & \qquad + \delta_{mp}\Big(2\gamma^{[ki]} \otimes \gamma^k + 2\gamma^k \otimes \gamma^{[ki]} - 2\gamma^i \otimes \mathbb{1} + 2\mathbb{1} \otimes \gamma^i\Big)\Big\} \end{aligned} \tag{B.32}$$

where we have used identity (B.16) to put the δ_{mp} terms in a simpler form.

B.5 Four free indices

Finally, we will need a single Fierz identity with four free indices, in order to eliminate terms at $1/r^7$ with two $\not\eta$s and an F_{ij}. The required identity is given by $v_2^n(f_n+g_n)=0$, with

$$\begin{aligned} f_n = & \; r_m r_p \mathrm{Tr}\left(L\gamma^i\gamma^m\gamma^{[a_1\ldots a_n]}\gamma^j F_{ij} L \; L\gamma^{[a_n\ldots a_1]}\gamma^p L\right) + \\ & + r_m r_p \mathrm{Tr}\left(L\gamma^i\gamma^{[a_1\ldots a_n]}\gamma^m\gamma^j F_{ij} L \; L\gamma^p\gamma^{[a_n\ldots a_1]} L\right), \end{aligned}$$

and

$$\begin{aligned} g_n = & \; r_m r_p \mathrm{Tr}\left(L\gamma^p\gamma^{[a_1\ldots a_n]}\gamma^m\gamma^i L F_{ij} L\gamma^j\gamma^{[lk]} L\right) + \\ & + r_m r_p \mathrm{Tr}\left(L\gamma^{[a_1\ldots a_n]}\gamma^i L F_{ij} L\gamma^j\gamma^m\gamma^{[a_n\ldots a_1]}\gamma^p L\right). \end{aligned}$$

Writing this identity explicitly with all gamma matrices appearing in antisymmetric products, we have (scaling by $1/4$ and ignoring $r\cdot K$ terms),

$$\begin{aligned} 0 = & \; r_m r_p \mathrm{Tr}\left(L * F_{ij} L \; L * L\right)\Big\{ -\frac{1}{2}\gamma^{[klmij]}\otimes\gamma^{[lkp]} - 2\gamma^{[mij]}\otimes\gamma^p + \gamma^m\otimes\gamma^{[pij]} - \\ & \qquad - \gamma^{[kmij]}\otimes\gamma^{[kp]} + \delta_{mp}(-\gamma^k\otimes\gamma^{[kij]} + \mathbb{1}\otimes\gamma^{[ij]})\Big\} + \\ & + r_m r_p \mathrm{Tr}\left(L * L F_{ij} L * L\right)\Big\{ 2\gamma^{[lmi]}\otimes\gamma^{[lpj]} + 2\gamma^{[mij]}\otimes\gamma^p + 2\gamma^m\otimes\gamma^{[pij]} - \\ & \qquad - 2\gamma^{[mi]}\otimes\gamma^{[pj]} + \delta_{mp}\Big(2\gamma^i\otimes\gamma^j - \gamma^{[kij]}\otimes\gamma^k - \\ & \qquad - \gamma^k\otimes\gamma^{[kij]} + \gamma^{[ij]}\otimes\mathbb{1} + \mathbb{1}\otimes\gamma^{[ij]}\Big)\Big\}, \end{aligned} \tag{B.33}$$

where we have used identity (B.29) to put the δ_{mp} terms in a simpler form.

References

[1] M. Claudson and M. Halpern, *Nucl. Phys.* **B 250** (1985) 689.

[2] R. Flume, *Ann. Phys. (NY)* **164** (1985) 189.

[3] M. Baake, P. Reinicke and V. Rittenberg, *J. Math. Phys.* **26** (1985) 1070.

[4] J. Goldstone, unpublished;
J. Hoppe, MIT Ph.D. thesis (1982); in *Proc. Int. Workshop on Constraint's theory and relativistic dynamics*, eds. G. Longhi and L. Lusanna, World Scientific, 1987.

[5] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Ann. Phys. (NY)* **185** (1988) 330.

[6] B. de Wit, J. Hoppe and H. Nicolai, *Nucl. Phys.* **B 305** (1988) 545.

[7] B. de Wit, M. Luscher and H. Nicolai, *Nucl. Phys.* **B 320** (1989) 135.

[8] T. Banks, W. Fischler, S. Shenker and L. Susskind, *M theory as a Matrix model: a conjecture, Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[9] M.R. Douglas, D. Kabat, P. Pouliot and S. Shenker, *D-branes and short distances in string theory, Nucl. Phys.* **B 485** (1997) 85 [hep-th/9608024].

[10] T. Banks, *Matrix theory, Nucl. Phys.* **67** *(Proc. Suppl.)* (1998) 180 [hep-th/9710231].

[11] D. Bigatti and L. Susskind, *Review of matrix theory,* hep-th/9712072.

[12] A. Bilal, *M(atrix) theory: a pedagogical introduction, Fortsch. phys.* **47** (1999) 5 [hep-th/9710136].

[13] W. Taylor, *Lectures on D-branes, gauge theory and M(atrices),* proceedings of Trieste summer school 1997, to appear, hep-th/9801182.

[14] S. Sethi and M. Stern, *D-brane bound states redux, Comm. Math. Phys.* **194** (1998) 675 [hep-th/9705046].

[15] O.J. Ganor, S. Ramgoolam and W. Taylor, *Branes, fluxes and duality in M(atrix)-theory, Nucl. Phys.* **B 492** (1997) 191 [hep-th/9611202].

[16] T. Banks, N. Seiberg and S. Shenker, *Branes from matrices, Nucl. Phys.* **B 490** (1997) 91 [hep-th/9612157].

[17] K. Becker, M. Becker, J. Polchinski and A. Tseytlin, *Higher order graviton scattering in M(atrix) theory, Phys. Rev.* **D 56** (1997) 3174 [hep-th/9706072].

[18] S. Paban, S. Sethi and M. Stern, *Constraints from extended supersymmetry in quantum mechanics, Nucl. Phys.* **B 534** (1998) 137 [hep-th/9805018].

[19] S. Paban, S. Sethi and M. Stern, *Supersymmetry and higher-derivative terms in the effective action of Yang-Mills theories, J. High Energy Phys.* **06** (1998) 012 [hep-th/9806028].

[20] P. Kraus, *Spin-Orbit interaction from Matrix theory, Phys. Lett.* **B 419** (1998) 73 [hep-th/9709199].

[21] J. Morales, C. Scrucca and M. Serone, *Scale independent spin effects in D-brane dynamics, Nucl. Phys.* **B 534** (1998) 223 [hep-th/9801183].

[22] I.N. McArthur, *Higher order spin-dependent terms in D0-brane scattering from the matrix model, Nucl. Phys.* **B 534** (1998) 183 [hep-th/9806082].

[23] M. Barrio, R. Helling and G. Polhemus, *Spin-spin interaction in M(atrix) theory, J. High Energy Phys.* **05** (1998) 012 [hep-th/9801189].

[24] J. Plefka, M. Serone and A. Waldron, *D = 11 SUGRA as the low energy effective action of Matrix Theory: three form scattering, J. High Energy Phys.* **11** (1998) 010 [hep-th/9809070].

[25] M. Dine and A. Rajaraman, *Multigraviton scattering in the matrix model, Phys. Lett.* **B 425** (1998) 77 [hep-th/9710174].

[26] M. Fabbrichesi, G. Ferretti and R. Iengo, *Supergravity and matrix theory do not disagree on multi-graviton scattering, J. High Energy Phys.* **06** (1998) 002 [hep-th/9806018].

[27] W. Taylor and M. Van Raamsdonk, *Three-graviton scattering in Matrix theory revisited, Phys. Lett.* **B 438** (1998) 248 [hep-th/9806066]; *Nucl. Phys.* **B 532** (1998) 227 [hep-th/9712159].

[28] Y. Okawa and T. Yoneya, *Multibody interactions of D-particles in supergravity and matrix theory, Nucl. Phys.* **B 538** (1999) 67 [hep-th/9806108].

[29] M. Dine, R. Echols and J.P. Gray, *Tree level supergravity and the matrix model,* hep-th/9810021.

[30] D. Kabat and W. Taylor, *Linearized supergravity from Matrix theory, Phys. Lett.* **B 426** (1998) 297 [hep-th/9712185].

[31] M. Berkooz and M.R. Douglas, *Five-branes in M(atrix) Theory, Phys. Lett.* **B 395** (1997) 196 [hep-th/9610236].

[32] M. Billó, P. Di Veccia, M. Frau, A. Lerda, R. Russo and S. Sciuto, *The Lorentz Force between D0 and D6 branes in string and M(atrix) theory, Mod. Phys. Lett.* **A 13** (1998) 2977 [hep-th/9805091].

[33] I. Chepelev and A. Tseytlin, *Long-distance interactions of branes: correspondence between supergravity and super Yang-Mills descriptions, Nucl. Phys.* **B 515** (1998) 73 [hep-th/9709087].

[34] E. Keski-Vakkuri and P. Kraus, *Born-Infeld actions from Matrix theory, Nucl. Phys.* **B 518** (1998) 212 [hep-th/9709122].

[35] V. Balasubramanian, R. Gopakumar and F. Larsen, *Gauge theory, geometry and the large N limit, Nucl. Phys.* **B 526** (1998) 415 [hep-th/9712077].

[36] E. Bergshoeff, M. Rakowski and E. Sezgin, *Phys. Lett.* **B 185** (1986) 371.

[37] R.R. Metsaev and M.A. Rahmanov, *Phys. Lett.* **B 193** (1987) 202.

[38] D. Kabat and W. Taylor, *Spherical membranes in Matrix theory, Adv. Theor. Math. Phys.* **2** (1998) 181 [hep-th/9711078].

[39] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring, Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[40] W. Taylor and M. Van Raamsdonk, *Angular momentum and long-range gravitational interactions in Matrix theory, Nucl. Phys.* **B 532** (1998) 227 [hep-th/9712159].

JHEP04(1999)013

[41] S. Hellerman and J. Polchinski, *Compactification in the lightlike limit, Phys. Rev.* **D 59** (1999) 125002 [hep-th/9711037].

[42] M. Van Raamsdonk, *Conservation of supergravity currents from matrix theory, Nucl. Phys.* **B 542** (1999) 262 [hep-th/9803003].

[43] J. Castelino, S. Lee and W. Taylor, *Longitudinal 5-branes as 4-spheres in Matrix theory, Nucl. Phys.* **B 526** (1998) 334 [hep-th/9712105].

[44] G. Lifschytz, *Four-brane and six-brane interactions in M(atrix) theory, Nucl. Phys.* **B 520** (1998) 105 [hep-th/9612223].

[45] V. Balasubramanian and F. Larsen, *Relativistic brane scattering, Nucl. Phys.* **B 506** (1997) 61 [hep-th/9703039].

[46] P.-M. Ho, M. Li and Y.-S. Wu, *p-p' strings in M(atrix) theory, Nucl. Phys.* **B 525** (1998) 146 [hep-th/9706073].

[47] E. Keski-Vakkuri and P. Kraus, *Notes on branes in Matrix theory, Nucl. Phys.* **B 510** (1998) 199 [hep-th/9706196].

[48] J.M. Pierre, *Comparing D-branes and black holes with 0- and 6-brane charge, Phys. Rev.* **D 56** (1997) 6710 [hep-th/9707102].

[49] V. Balasubramanian, D. Kastor, J. Traschen and K.Z. Win, *The spin of the M2-brane and spin-spin interactions via probe techniques*, *Phys. Rev.* **D 59** (1999) 084007 [hep-th/9811037].

[50] P. Van Nieuwenhuizen, *Phys. Rep.* **60** (1981) 189.

[51] D. Lowe, *Constraints on higher derivative operators in the Matrix theory effective lagrangian, J. High Energy Phys.* **11** (1998) 009 [hep-th/9810075].

[52] I. Chepelev and A. Tseytlin, *On membrane interaction in Matrix theory, Nucl. Phys.* **B 524** (1998) 69 [hep-th/9801120].

[53] A.A. Tseytlin, *On non-abelian generalisation of Born-Infeld action in string theory, Nucl. Phys.* **B 501** (1997) 41 [hep-th/9701125].

[54] R.R. Metsaev and A.A. Tseytlin, *Nucl. Phys.* **B 298** (1988) 109.

[55] J. Maldacena, *The large N limit of superconformal theories and supergravity, Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[56] M. Dine and N. Seiberg, *Comments on higher derivative operators in some SUSY field theories, Phys. Lett.* **B 409** (1997) 239 [hep-th/9705057].

[57] D. Lowe and R. von Unge, *Constraints on higher derivative operators in maximally supersymmetric gauge theory, J. High Energy Phys.* **11** (1998) 014 [hep-th/9811017].

[58] Y. Okawa and T. Yoneya, *Equations of motion and Galilei invariance in D-particle dynamics, Nucl. Phys.* **B 541** (1999) 163 [hep-th/9808188].

[59] M.R. Douglas, H. Ooguri and S. Shenker, *Issues in M(atrix) theory compactification, Phys. Lett.* **B 402** (1997) 36 [hep-th/9702203].

[60] M.R. Douglas, *D-branes in curved space,* hep-th/9703056.

[61] M.R. Douglas, *D-branes and matrix theory in curved space, Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 381 [hep-th/9707228].

[62] M.R. Douglas, A. Kato and H. Ooguri, *D-brane actions on Kahler manifolds,* hep-th/9708012.

[63] N. Seiberg, *Why is the Matrix model correct?, Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009].

[64] M.R. Douglas and H. Ooguri, *Why Matrix theory is hard, Phys. Lett.* **B 425** (1998) 71 [hep-th/9710178].

[65] G. Lifschytz, *DLCQ-M(atrix) description of string theory and supergravity, Nucl. Phys.* **B 534** (1998) 83 [hep-th/9803191].

[66] E. Cremmer and S. Ferrara, *Phys. Lett.* **B 91** (1980) 61.

[67] B. de Wit, K. Peeters and J. Plefka *Superspace geometry for supermembrane backgrounds, Nucl. Phys.* **B 532** (1998) 99 [hep-th/9803209].

[68] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[69] W. Taylor and M. Van Raamsdonk, *Multiple D0-branes in weakly curved backgrounds,* hep-th/9904095.

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